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Journal of Economic Theory 146 (2011) 2596–2607

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

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Optimal advertising of auctions ☆

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Received 13 July 2010; final version received 17 April 2011; accepted 25 June 2011

Available online 14 October 2011

Abstract

We study a symmetric independent private values auction model where the revenue-maximizing seller faces a cost c_n of attracting n bidders to the auction. If the distribution of valuations possesses an increasing failure rate (IFR), the seller overinvests in attracting bidders compared to the social optimum. Conversely, if the distribution is DFR, the seller underinvests compared to the social optimum. If the distribution of valuations becomes more dispersed, both, a revenue- and a welfare-maximizing seller, attract more bidders.

JEL classification: D44

Keywords: Auctions; Advertising; Order statistics

1. Introduction

We analyze a symmetric independent private values auction model: a revenue-maximizing seller faces a cost c_n of attracting n bidders. These costs can be thought of as advertising costs – or as costs of making bidders familiar with the object to be auctioned. We mainly consider the question: How many bidders does the seller choose to attract compared to the socially optimal number?

☆ I would like to thank Martin Hellwig, Johannes Koenen, Benny Moldovanu, and Patrick Schmitz, as well as the associate editor and two referees. The paper first circulated under the title "Information Release in Second Price Auctions".

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0022-0531/$ – see front matter
doi:10.1016/j.jet.2011.10.010

Our main result is the following: If the distribution of valuations has an increasing failure rate (IFR), the seller overadvertises the auction. Conversely, with a decreasing failure rate (DFR), the seller underadvertises. The difference in investment behavior stems from the fact that in the IFR case bidders' aggregate rents, i.e., the difference between the two highest valuations, decrease in the number of bidders. Conversely, they increase in the DFR case. Therefore, the bidders' aggregate rents determine whether the seller over- or underadvertises when maximizing revenues, compared to the social optimum.

We first develop the results for standard auctions, e.g., a second price auction without a reserve price. Then we demonstrate that they generalize to optimal auctions with minor caveats. Moreover, we show that in an optimal auction the seller advertises to fewer bidders than in a standard auction. The reason for this is that a reserve price is more effective with fewer bidders. Thus, smaller numbers of bidders are comparatively more profitable in an optimal auction than in a standard auction without reserve. Finally, we show that, under a more dispersed distribution of valuations, both revenue- and welfare-maximizing sellers increase their advertising efforts. Technically, we rely on tools from reliability theory which allow us to derive unambiguous results for broad classes of probability distributions.[1]

The advertising literature frequently assumes costs of attracting prospective buyers.[2] In the auction literature, such costs have received surprisingly little attention despite the fact that they are often implicitly assumed: For example, consider Bulow and Klemperer's [4] result that attracting another bidder is more profitable than setting an optimal reserve price. In their comparison, they implicitly assume that it is costly to set the optimal reserve or to attract one more bidder: If there were no costs associated with attracting more bidders, the comparison would be trivial.

Our results are related to models in which bidders strategically decide about entry to an auction such as in French and McCormick [7], McAfee and McMillan [11], and Levin and Smith [9].[3] In this literature, it is assumed that the bidders (and not the seller) face costs of entering the auction. Both, McAfee and McMillan [11], and Levin and Smith [9], consider a revenue-maximizing seller who can influence the number of entering bidders, respectively, by entry fees or by directly restricting the pool of entrants. Both papers find that the seller's incentives coincide with social incentives. The reason is that the bidders in these models decide about entering the auction before they learn their valuations. Hence bidders end up earning zero expected profits.[4] In line with our findings, the monotonicity of the bidders' aggregate rents is crucial: by the fact that the bidders' aggregate rents are constant (zero), the revenue-maximizing seller's interests are aligned with those of a welfare-maximizer. This coincidential alignment of incentives stands in marked contrast to the findings in our model.

Our technical results also contribute to a recent literature which tries to develop a better understanding of Myerson's [13] regularity condition of increasing virtual valuations.[5] We provide two results in this context: First, we show that increasing virtual valuations are

[1] See, e.g., Barlow and Proschan [2]. For previous applications of reliability theory to the auction literature, see Li [10] and Moldovanu, Sela and Shi [12], and the references therein.

[2] See, e.g., Bagwell [1].

[3] See Bergemann and Välimäki [3] for a recent survey which covers this literature.

[4] In McAfee and McMillan [11], the seller extracts all rents via ex-ante entry fees. Levin and Smith consider a symmetric equilibrium with mixing over the entry decision. There, bidders set their entry probabilities such that they all earn zero expected payoffs.

[5] See Ewerhart [6] and the references therein.

linked to monotonicity in n of the sequence of increments of expected second order statistics $E[X_{2:n+1} - X_{2:n}]$. This connection parallels the connection between the IFR condition and monotonicity of $E[X_{1:n} - X_{2:n}]$. Notably, increasing virtual valuations are in a sense a tight condition for concavity of expected second order statistics.[6] Our proof relies on a recent observation by Ewerhart [6]: A distribution F with density f possessing increasing virtual valuations is equivalent to F possessing an increasing zoom rate

$$z_F(x) = \frac{f(x)}{(1 - F(x))^2}.$$

Second, we find that (strictly) increasing virtual valuations are not sufficient for our purposes and for optimal auctions when considering distributions of valuations with an unbounded support: We give an example of a distribution with strictly increasing virtual valuations for which no optimal reserve exists.

The paper proceeds as follows: Section 2 introduces the model and the optimization problems. Section 3 develops the technical tools needed for our analysis and proves concavity of first and second order statistics. Section 4 contains our results on standard auctions: We compare social and revenue-maximizing incentives for inviting bidders and study the influence of dispersion in the distribution of valuations. Section 5 extends our analysis to revenue-maximizing auctions. Section 6 concludes. All proofs are in Appendix A.

2. The model

We consider a standard symmetric independent private values auction model with a seller who sells an indivisible object to a group of bidders. The bidders' valuations X_i are independent draws from a distribution F. We denote by $X_{k:n}$ the kth largest of the random variables $X_1, \ldots, X_n$ and assume that $E[X_i] < \infty$.[7] There is an infinite pool of potential bidders who are initially unaware of the auction. The seller has to invest c_n to make n bidders aware of the auction. Once a bidder i becomes aware of the auction, he privately learns his valuation X_i for the object for sale. The seller values the object at zero. The object is auctioned off in a sealed-bid second price auction with reserve price r. We assume throughout that bidders adhere to their weakly dominant strategy of bidding their valuation whenever it is weakly greater than the reserve price.

We assume that the cost sequence c_n is weakly convex and strictly increasing for $n \geqslant 1$. Moreover, we assume $c_0 = c_1 = 0$, i.e., the seller can attract one bidder for free.[8] We assume that the distribution F possesses a continuous density f which is strictly positive over the support S of F where $S = [0, s)$ for some $s \in (0, \infty]$.[9] We assume that F fulfills Myerson [13]'s regularity condition that the virtual valuation function V_F,

$$V_F(x) = x - \frac{1 - F(x)}{f(x)},$$

is strictly increasing in x. Moreover, we assume the following:

6 This concavity is important in our analysis since it guarantees that the seller's maximization problem is well behaved.
7 The latter assumption ensures that all order statistics of F have finite expectation: $E[X_{k:n}] < nE[X_1] < \infty$.
8 As will become clear below, this assumption allows us to avoid a separate discussion of the case $n = 1$. We can easily relax it to the assumption that costs grow slowly enough such that the relevant range of n is above 1.
9 These regularity assumptions are made to avoid technicalities and can easily be relaxed, e.g., to densities which are zero on some intervals.

A1 There exists an $r^* < \infty$ such that $V_F(r^*) = 0$.
A2 The zoom rate Z_F defined by

$$Z_F(x) = \frac{f(x)}{(1 - F(x))^2}$$

converges to ∞ as x gets large.

Assumptions A1 and A2 lead to a mild but non-trivial strengthening of the increasing virtual valuations condition. They ensure that F is not only Myerson regular but also not on the border of Myerson regularity. A1 is equivalent to assuming the existence of an optimal reserve price.[10] Since $V_F(0) = -\frac{1}{f(0)} \leqslant 0$, increasing virtual valuations together with A1 guarantee that virtual valuations increase strongly enough to become positive from some point on. For distributions on $\mathbb{R}^+$, increasing virtual valuations alone are not sufficient to guarantee this.[11] Assumption A2 is related to the increasing virtual valuations condition as stated by Ewerhart [6]: A distribution F has an increasing zoom rate Z_F iff V_F is increasing.[12] In this sense, A2 is – just like A1 – an assumption of sufficient growth of virtual valuations.

Our main goal is to compare the optimal choice of n under three different objectives: a) Maximizing social welfare in a second price auction with reserve price 0,[13] b) maximizing the seller's revenue in a second price auction with reserve price 0 and c) maximizing revenue in a second price auction with reserve price chosen by the seller. The decision problem a) of a welfare-maximizing seller is given by

$$\max_n E[X_{1:n}] - c_n,$$

i.e., the seller maximizes the valuation of the winning bidder minus the invitation costs. In the following, we denote by n_w a solution to this maximization problem.

The decision problem b) of a revenue-maximizing seller who sets a reserve price of zero is given by

$$\max_n E[X_{2:n}] - c_n,$$

since the second-highest valuation is the price paid by the winning bidder. Denote by n_p a solution to this optimization problem.

Finally, the decision problem c) of a revenue-maximizing seller who sets the optimal reserve price r^* is given by

$$\max_n o_n - c_n \quad \text{where } o_n = E\big[X_{2:n}\mathbb{1}_{\{X_{2:n} \geqslant r^*\}} + r^*\mathbb{1}_{\{X_{1:n} \geqslant r^* > X_{2:n}\}}\big],$$

since the seller's revenue is then given by r^* if only one bidder has a valuation above r^* and by the second-highest valuation if at least two bidders have a valuation above r^*. As shown by Myerson [13] the revenue-optimal reserve price is the solution of

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10 See, e.g., Krishna [8].

11 For an example consider the distribution function $F(x) = \frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}}$ on $\mathbb{R}^+$. This distribution possesses strictly increasing virtual valuations yet any finite reserve price is dominated by all larger reserve prices.

12 For a quick verification, note that both conditions correspond to the same first order condition, $2f(x)^2 + (1 - F(x))f'(x) > 0$. Ewerhart [6] shows the equivalence under considerably weaker regularity conditions on f.

13 Setting the reserve price to zero is obviously the welfare-maximizing choice.

$$r^* = \frac{1 - F(r^*)}{f(r^*)}.$$

A solution exists by Assumption A1 and is unique since virtual valuations are increasing. As shown by Myerson [13], o_n can also be written as

$$o_n = E\big[\max\big(V_F(X_{1:n}), 0\big)\big].$$

In the following sections we study the ranking of the numbers of bidders attracted to the auction, n_w, n_p and n_o, under different assumptions on F. If F is common knowledge and the bidders are aware of the revenue-maximizing seller's choice of n, maximization problem b) is equivalent to the corresponding problems for all standard auctions by the revenue-equivalence theorem. Likewise, maximization problem c) is equivalent to the corresponding problem for all revenue-maximizing mechanisms. Common knowledge of n can arise for example if the bidders can observe n during the auction, if the seller can credibly announce n, or if the bidders can infer the seller's choice of n from his optimization problem.

Accordingly, if n is observable, the welfare-maximization problem a) is equivalent to the problem of a revenue-maximizing seller who charges entry fees before the bidders observe their valuations. The problem c) of maximizing $o_n - c_n$ is equivalent to the problem of a seller who charges an entry fee which the bidders pay after they have observed their valuations.

3. Technical prerequisites

The following two technical observations form the basis of our analysis: First, extremal order statistics are easy to control and, second, many interesting quantities can be expressed as extremal order statistics. The next lemma establishes the first of these observations.

Lemma 1. *Let $X_1, X_2, \ldots$ be the sequence of valuations introduced above.*

(i) *Let h be a weakly increasing, non-negative function with $E[h(X_1)] < \infty$ and for which $h(X_1)$ is not almost surely constant. Then $E[h(X_{1:n})]$ is a strictly increasing and strictly concave sequence. Moreover, if $\lim_{x\to s} h(x) = \infty$, then $\lim_{n\to\infty} E[h(X_{1:n})] = \infty$ where, as before, s denotes the supremum of the support of the X_i.*

(ii) *Let h be a weakly decreasing, non-negative function with $E[h(X_1)] < \infty$ and for which $h(X_1)$ is not almost surely constant. Then $E[h(X_{1:n})]$ is a strictly decreasing and strictly convex sequence. Moreover, if $\lim_{x\to s} h(x) = 0$, then $\lim_{n\to\infty} E[h(X_{1:n})] = 0$.*

The idea behind Lemma 1 is that for an increasing function h the random variable $h(X_{1:n})$ is the first order statistic of the random variables $(Y_i)_i = (h(X_i))_i$ while for a decreasing function h the random variable $h(X_{1:n})$ is the lowest order statistic of the random variables $(Y_i)_i = (h(X_i))_i$.

From the lemma, we can immediately conclude that $E[X_{1:n}]$ and o_n are increasing and concave sequences by setting $h(x) = x$ and $h(x) = \max(V_F(x), 0)$, respectively: Both choices of h are increasing. Our first main result shows that under increasing virtual valuations the sequence $E[X_{2:n}]$ is concave as well. Moreover, it shows that the assumption of increasing virtual valuations is, in a sense, a tight condition for the concavity of $E[X_{2:n}]$.

Lemma 2. *It holds that*

$$E[X_{2:n+1} - X_{2:n}] = E\big[h(X_{1:n})\big]$$

where

$$h(x) = \frac{(1 - F(x))^2}{f(x)}.$$

Accordingly, under our assumptions on F, the sequence $E[X_{2:n}]$ is strictly concave and its increments go to zero as n increases.

Note that the crucial observation here is not that we can rewrite $E[X_{2:n+1} - X_{2:n}]$ in terms of $X_{1:n}$. Rather, it is that n is independent of the function h which makes the further analysis possible: Provided that h is monotone, we can now analyze $E[X_{2:n+1} - X_{2:n}]$ as a sequence of extremal order statistics. Moreover, recall that strictly increasing virtual valuations are equivalent to a strictly increasing zoom rate Z_F and accordingly equivalent to a strictly decreasing function h in Lemma 2. The strict concavity of $E[X_{2:n}]$ is thus a consequence of the increasing virtual valuations condition while the fact that the increments go to zero follows from Assumption A2.

From these concavity results we can conclude that our maximization problems are sufficiently well behaved:

Corollary 1.

(i) *If n_w maximizes $E[X_{1:n}] - c_n$, then $n_w + 2$ does not maximize $E[X_{1:n}] - c_n$.*
(ii) *If n_p maximizes $E[X_{2:n}] - c_n$, then $n_p < \infty$ and $n_p + 2$ does not maximize $E[X_{2:n}] - c_n$.*

The corollary shows that maximizers are almost unique[14] and that – due to the finiteness of n_p – one can always meaningfully compare n_p to n_w. In Section 5 we prove that $n_o \leqslant n_p$ which settles the corresponding question for optimal auctions.

Conversely, distributions F which exhibit strictly decreasing virtual valuations possess a strictly convex sequence $E[X_{2:n}]$ provided that $E[X_{2:n}]$ is finite.[15] This shows that the increasing virtual valuations condition is in a sense a tight condition for concavity of second order statistics. This appears to be a novel observation.

4. Standard auctions

In this section, we compare the number of bidders attracted by a revenue-maximizing seller in a standard auction, n_p, to the socially optimal number of bidders n_w. Furthermore, we show that more bidders get attracted for more dispersed distributions of valuations.

4.1. Over- and underadvertising

The comparison of the numbers of invited bidders is based on the following lemma. A revenue-maximizing seller overadvertises if his revenue $E[X_{2:n}]$ reacts more strongly to the number of bidders than welfare $E[X_{1:n}]$ and vice versa.

[14] Due to the discrete character of the problem, uniqueness is generically fulfilled but hard to guarantee – it is easy to construct examples where two subsequent values of n are optimizers.

[15] For examples of such distributions consider the power law distributions with density $f_\gamma(x) = (\gamma - 1)(1 + x)^{-\gamma}$ for $\gamma \in (1.5, 2)$.

Lemma 3.

(i) *If* $E[X_{1:n} - X_{2:n}]$ *is strictly increasing, it holds that* $n_p \leqslant n_w$. *Hence a revenue-maximizing seller invites less bidders than in the social optimum.*
(ii) *If* $E[X_{1:n} - X_{2:n}]$ *is strictly decreasing, it holds that* $n_p \geqslant n_w$. *Hence a revenue-maximizing seller invites more bidders than in the social optimum.*

The lemma is based on the following equivalence:

$$E[X_{1:n+1} - X_{2:n+1}] > E[X_{1:n} - X_{2:n}] \quad \Leftrightarrow \quad E[X_{1:n+1} - X_{1:n}] > E[X_{2:n+1} - X_{2:n}]$$

for all n. Thus, if $E[X_{1:n} - X_{2:n}]$ is increasing, the gains from attracting another bidder are larger with regard to social welfare than with regard to the seller's revenue.

We next identify conditions determining the monotonicity behavior of $E[X_{1:n}] - E[X_{2:n}]$. For this purpose, we apply the following result from reliability theory[16] which is also an immediate consequence of Lemma 1:

Lemma 4. *It holds that*

$$E[X_{1:n} - X_{2:n}] = E\big[h(X_{1:n})\big] \quad \textit{where } h(x) = \frac{1 - F(x)}{f(x)}.$$

Accordingly, $E[X_{1:n}] - E[X_{2:n}]$ *is strictly increasing if* h *is increasing and strictly decreasing if* h *is decreasing.*

The function h in the lemma is the inverse of the failure rate H_F of F which is defined by

$$H_F(x) = \frac{f(x)}{1 - F(x)}.$$

Distributions for which H_F is increasing or decreasing are known, respectively, as IFR and DFR distributions. Putting these observations together we obtain the following version of Lemma 3, which is the main result of this section:

Proposition 1.

(i) *If* F *is DFR, it holds that* $n_p \leqslant n_w$. *Hence the seller underadvertises.*
(ii) *If* F *is IFR, it holds that* $n_p \geqslant n_w$. *Hence the seller overadvertises.*

The distinction between IFR and DFR is crucial for the tail behavior of F: The boundary case between IFR and DFR is the exponential distribution which has a constant failure rate. Here (and only here), revenue-maximizing and social incentives for attracting bidders are aligned. IFR distributions form a class of distributions with lighter than exponential tails. For them, the second order statistic reacts more sensitively to changes in the number of bidders than the first order statistic. The converse holds for DFR distributions, which are more heavy-tailed than the exponential distribution.[17]

[16] See, e.g., Barlow and Proschan [2].

[17] Examples of IFR distributions are, for instance, Gaussian distributions (restricted to $\mathbb{R}^+$) and many distributions with finite support such as uniform distributions. The power-law distributions are examples of DFR distributions.

4.2. Dispersion

Let us now study how n_w and n_p react to changes in the dispersion of the distribution of the bidders' valuations. For this purpose, we stress the dependence of n_w, n_p, $E[X_{1:n}]$, and $E[X_{2:n}]$ on F by writing n_w^F, n_p^F, $E[X_{1:n}^F]$, and $E[X_{2:n}^F]$.

We compare the optimal advertising levels under different distributions of valuations which are ordered in the dispersive order.[18] A distribution F is said to dominate a distribution G in the dispersive order if for all $0 \leqslant a < b \leqslant 1$

$$F^{-1}(b) - F^{-1}(a) \geqslant G^{-1}(b) - G^{-1}(a),$$

i.e., if the distance between any pair of quantiles is larger under F than under G. This order is well suited for our analysis since it allows us to control the behavior of increments of order statistics.[19]

Expectations of order statistics – which are closely related to quantiles – lie further apart under a more dispersed distribution of valuations, see, e.g., Theorem 3.B.31 of Shaked and Shantikumar [14]. Hence both the revenue-maximizer and the welfare-maximizer attract more bidders the more dispersed the distribution of the bidders' valuations is:

Proposition 2. *Consider two distributions of valuations fulfilling the assumptions of our model where F dominates G in the dispersive order. Then the sequences $E[X_{1:n}^F] - E[X_{1:n}^G]$ and $E[X_{2:n}^F] - E[X_{2:n}^G]$ are increasing in n. Therefore, it holds that $n_w^F \geqslant n_w^G$ and $n_p^F \geqslant n_p^G$.*

4.3. Relaxing Assumptions A1 and A2

All results of this section still hold if we drop Assumptions A1 and A2, require only non-decreasing virtual valuations and impose additional assumptions on the cost sequence c_n to ensure finiteness of n_p. For instance, one could assume that c_n is strictly convex and that its increments become arbitrarily large as n goes to infinity. Such a generalization is not possible for the results of the next section which require the existence of a finite optimal reserve price.

5. Optimal auctions

We now turn to the optimization problem of a revenue-maximizing seller who can set an optimal reserve price. Recall that we denote the sequence of expected revenues o_n and the optimal number of bidders n_o. Since it holds that $E[X_{2:n}] < o_n < E[X_{1:n}]$, let us consider the increments $E[X_{1:n}] - o_n$ and $o_n - E[X_{2:n}]$ in the next lemma. The monotonicity behavior of $E[X_{1:n}] - o_n$ determines whether over- or underinvestment occurs:

Lemma 5. *It holds that*

$$E[X_{1:n}] - o_n = E\big[h_1(X_{1:n})\big] \quad \text{where } h_1(x) = \min\left(x, \frac{1 - F(x)}{f(x)}\right)$$

[18] See Shaked and Shantikumar [14] for more background.

[19] Many weaker dispersion criteria such as F having a larger variance than G would not suffice for this purpose.

and

$$o_n - E[X_{2:n}] = E\big[h_2(X_{1:n})\big] \quad \text{where } h_2(x) = \left(\frac{1-F(x)}{f(x)} - x\right)\mathbb{1}_{\{x<r^*\}}.$$

By the increasing virtual valuations assumption, the function h_2 is decreasing and converges to zero. Accordingly, we obtain:

Corollary 2. *The sequence $o_n - E[X_{2:n}]$ is decreasing and converges to zero. Hence a revenue-maximizing seller who sets an optimal reserve advertises less than a revenue-maximizing seller who cannot set a reserve*: $n_o \leqslant n_p$.

The comparison of n_o to the welfare-optimal choice n_w is more subtle: If F is DFR, then h_1 is increasing so that our underinvestment result also holds for optimal reserve prices:

Corollary 3. *If F is DFR, then $E[X_{1:n}] - o_n$ is increasing. Hence the revenue-maximizing seller underadvertises*: $n_o \leqslant n_w$.

Under IFR, the function h_1 is first increasing and then decreasing: It increases linearly until r^* and then decreases as it equals the inverse failure rate. Since increasing n moves the distribution of $X_{1:n}$ further into the right tail, the decreasing part of h_1 typically dominates for sufficiently large n. Hence, for large n, the reserve price plays a negligible role. Accordingly, we can expect to observe overinvestment under IFR distributions from some n on. We expect this effect to be more pronounced for distributions with a strongly increasing failure rate. We confirm this intuition with three examples: The exponential distribution which has a constant failure rate, the uniform distribution which is strongly IFR, and finally two distributions which are IFR but close to the exponential distribution.

Example 1. If F is the exponential distribution which lies at the boundary between IFR and DFR behavior, we know that $E[X_{1:n}] - E[X_{2:n}]$ is constant. Since $o_n - E[X_{2:n}]$ decreases in n, the remainder $E[X_{1:n}] - o_n$ must increase. Thus the exponential distribution is no longer a boundary case – it behaves just like a DFR distribution. Accordingly, we observe underinvestment.

Example 2. If F is the uniform distribution on [0, 1], and thus a distribution without tails which is "strongly" IFR, $E[X_{1:n}] - o_n$ is decreasing. Here, the decreasing part of h_1 is powerful enough to always dominate the increasing part.[20] Accordingly, for the uniform distribution a revenue-maximizer conducting an optimal auction overinvests.

Example 3. Consider the distribution F with density $f(x) = x\exp(-x)$. This distribution has the same tail behavior as the exponential distribution but it is strictly IFR. Here, the sequence $E[X_{1:n}] - o_n$ is increasing in n until $n^* = 8$ and decreases from there on: As predicted above, the IFR behavior takes over at some point. This happens despite the fact that F behaves essentially like the exponential distribution for large x. The same behavior with $n^* = 3$ is observed for Gaussian distributions restricted to $\mathbb{R}^+$. In these examples, overinvestment occurs if marginal costs are low enough to guarantee that the relevant range of n is sufficiently high.

[20] In this case, h_1 is the symmetric function $h_1 = \min(x, 1-x)$ on [0, 1]. Since $X_{1:n}$ has more mass on [0.5, 1] than on [0, 0.5] for $n > 1$, the decreasing part of h_1 is dominant for all n.

6. Conclusion

We have studied a symmetric independent private values auction model in which the revenue-maximizing seller advertises the auction to the bidders. Our main results show that the failure rate determines whether the seller over- or underadvertises compared to the social optimum. So far, we have mainly discussed our results in the context of auction theory. We would like to conclude by discussing their place in the advertising literature.

In a classical paper, Shapiro [15] demonstrates that a revenue-maximizing monopolist who cannot price-discriminate and who can sell any number of objects underprovides informative advertising. The reason is that he cannot extract the whole surplus from the consumers: He does not fully internalize the gains from advertising and selling to more consumers. In contrast, in our model, the product of the seller is scarce and the selling price is determined by the auction mechanism. The seller then underadvertises whenever the expected selling price in the auction reacts too little to advertising. Yet for many distribution functions, the seller overadvertises as the selling price reacts more strongly to advertising than the winning bidder's valuation and thus welfare. To our knowledge, our study is the first to capture both phenomena, over- and underadvertising, within one model.

Appendix A. Proofs

We denote the distribution of $X_{k:n}$ by $F_{k:n}$ and its density by $f_{k:n}$. Recall that[21]

$$F_{1:n}(x) = F(x)^n, \qquad f_{1:n}(x) = nF^{n-1}(x)f(x),$$
$$F_{2:n} = F(x)^n + nF^{n-1}(x)\big(1 - F(x)\big),$$

and $F_{n:n}(x) = 1 - (1 - F(x))^n$.

Proof of Lemma 1. Consider first independent, a.s. non-constant, positive random variables Y_i from a distribution G with finite expectation (but possibly with atoms). It holds that

$$E[Y_{1:n}] = \int_0^\infty 1 - G(x)^n\,dx \quad \text{and} \quad E[Y_{1:n+1} - Y_{1:n}] = \int_0^\infty \big(1 - G(x)\big)G(x)^n\,dx.$$

Since Y_1 is not almost surely constant, we have $G(x) \in (0,1)$ on an interval of positive mass. Thus the first integral is strictly increasing and the second one is strictly decreasing in n. This shows that expectations $E[Y_{1:n}]$ of first order statistics are monotonically increasing and concave. With an analogous argument, it follows that $E[Y_{n:n}]$ is decreasing and convex. Now we consider $E[h(X_{1:n})]$. For an increasing function h, it holds that $E[h(X_{1:n})] = E[Y_{1:n}]$ where we define $Y_i = h(X_i)$. For a decreasing h, it holds that $E[h(X_{1:n})] = E[Y_{n:n}]$. This shows the monotonicity, concavity and convexity properties of $E[h(X_{1:n})]$.

Next we show that an increasing h which converges to infinity implies that $E[h(X_{1:n})]$ converges to infinity. Fix some $m > 0$. We show that, from some sufficiently large n on, $E[h(X_{1:n})] > m$. By assumption, there is some $x^* \in (0, s)$ with $h(x^*) > 2m$. We can thus bound $E[h(X_{1:n})]$ by

[21] See David [5].

$$E\big[h(X_{1:n})\big] = \int_0^s h(x) f_{1:n}(x)\,dx \geqslant \int_{x^*}^s 2m f_{1:n}(x)\,dx \geqslant 2m\big(1 - F\big(x^*\big)^n\big).$$

Thus we can guarantee $E[h(X_{1:n})] > m$ by choosing n sufficiently large, implying that $E[h(X_{1:n})]$ converges to infinity. The proof for a decreasing h which converges to zero is completely analogous: For sufficiently large n, most mass of $F_{1:n}$ lies on values where h is small. □

Proof of Lemma 2. It holds that

$$\begin{aligned} E[X_{2:n+1} - X_{2:n}] &= \int_0^s F_{2:n}(x) - F_{2:n+1}(x)\,dx \\ &= \int_0^s n\big(F(x)^{n-1} - 2F(x)^n + F(x)^{n+1}\big)\,dx \\ &= \int_0^s nF(x)^{n-1}\big(1 - F(x)\big)^2\,dx = \int_0^s h(x) f_{1:n}(x)\,dx. \end{aligned}$$

h is decreasing by the zoom rate formulation of increasing virtual valuations and it converges to zero by A2. The results thus follow from Lemma 1. □

Proof of Corollary 1. By assumption, c_n is weakly convex and, by Lemma 1, the sequences $E[X_{1:n}]$ and $E[X_{2:n}]$ are strictly concave. Thus $a_n = E[X_{1:n}] - c_n$ and $b_n = E[X_{2:n}] - c_n$ are strictly concave. If n^* and $n^* + 2$ were maximizers of a_n, we would have $a_n^* = a_{n^*+2}$ which would imply $a_{n^*+1} > \max(a_{n^*}, a_{n^*+2})$. Thus at most two subsequent numbers n can be maximizers. The same is true for b_n. Moreover, since the increments of $E[X_{2:n}]$ converge to zero by Lemma 2 and since c_n increases at least linearly, b_n decreases from some point on. Thus we must have $n_p < \infty$. □

Proof of Lemma 3. If $E[X_{1:n}] - E[X_{2:n}]$ is increasing, it holds that $E[X_{1:n+1}] - E[X_{1:n}] > E[X_{2:n+1}] - E[X_{2:n}]$. Thus the value of n which balances gains and costs from attracting an additional bidder is larger under welfare-maximization than under revenue-maximization. This implies $n_w \geqslant n_p$. The case where $E[X_{1:n}] - E[X_{2:n}]$ is decreasing is analogous. □

Proof of Lemma 4. The lemma follows from

$$\begin{aligned} E[X_{1:n} - X_{2:n}] &= \int_0^s F_{2:n}(x) - F_{1:n}(x)\,dx = \int_0^s nF(x)^{n-1}\big(1 - F(x)\big)\,dx \\ &= \int_0^s h(x) f_{1:n}(x)\,dx \end{aligned}$$

and from Lemma 1. □

Proof of Proposition 1. The proposition is an immediate consequence of Lemma 3, Lemma 4 and the definitions of IFR and DFR. □

Proof of Proposition 2. Denote by $X_{k:n}$ and $Y_{k:n}$ the respective order statistics from F and G. From the recurrence relations on p. 45 of David [5] and from Theorem 3.B.31 of Shaked and Shantikumar [14], it follows that

$$E[X_{1:k} - X_{1:k-1}] = \frac{1}{k} E[X_{1:k} - X_{2:k}] \geqslant \frac{1}{k} E[Y_{1:k} - Y_{2:k}] = E[Y_{1:k} - Y_{1:k-1}]$$

and

$$E[X_{2:k} - X_{2:k-1}] = \frac{2}{k} E[X_{2:k} - X_{3:k}] \geqslant \frac{2}{k} E[Y_{2:k} - Y_{3:k}] = E[Y_{2:k} - Y_{2:k-1}].$$

Thus arguing as in the proof of Lemma 3 proves our claim. □

Proof of Lemma 5. Recall that

$$o_n = E\big[\max\big(V_F(X_{1:n}), 0\big)\big] = \int_{r^*}^{\infty} V_F(x) f_{1:n}(x)\, dx.$$

This implies the desired expression for $E[X_{1:n}] - o_n$. The expression for $o_n - E[X_{2:n}]$ follows from $o_n - E[X_{2:n}] = (E[X_{1:n}] - E[X_{2:n}]) - (E[X_{1:n}] - o_n)$ together with our expressions for $E[X_{1:n}] - E[X_{2:n}]$ and $E[X_{1:n}] - o_n$. □

Proof of Corollary 2. The properties of $o_n - E[X_{2:n}]$ follow directly from Lemma 1 and Lemma 5 as the function $h_2(x)$ from Lemma 5 is decreasing by the increasing virtual valuations condition and zero for $x > r^*$. Comparing the maximization problems for $E[X_{2:n}] - c_n$ and $o_n - c_n$ yields that the latter problem must have smaller solutions n_o since we add a decreasing sequence to the objective function of the former problem. □

Proof of Corollary 3. This follows from Lemma 1 and Lemma 5 as the function $h_1(x)$ from Lemma 5 is increasing if F is DFR. The inequality $n_w \geqslant n_o$ follows like in the proof of Lemma 3. □

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