Revenues and Welfare in Auctions with Information Release*

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Abstract

Auctions are the allocation-mechanisms of choice whenever goods and information in markets are scarce. Therefore, understanding how information affects welfare and revenues in these markets is of fundamental interest. We introduce new statistical concepts, k- and k-m-dispersion, for understanding the impact of information release. With these tools, we study the comparative statics of welfare versus revenues for auctions with one or more objects and varying numbers of bidders. Depending on which parts of a distribution of valuations are most affected by information release, welfare may react more strongly than revenues, or vice versa.

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1 Introduction

Auctions improve allocations in markets in which information is scarce. They are not only applied in highly capitalized spectrum and timber markets, but also in selling various items from used cars to fine arts. Auctions also serve as models of competition for prizes such as college admissions, winning districts in elections, or finishing among the first in R&D races. As they are known to allocate scarce goods well despite a fundamental lack of information about the bidders' costs or valuations, it is surprising how little we know about the interaction of information, welfare and revenues in these contexts. This is the starting point of our paper.

Generating information is typically a costly endeavor. For a welfare maximizer, the incentives to provide information on the goods for sale may be very different from the incentives a revenue-maximizing seller faces. The reason behind is that a welfare maximizer incorporates bidders' aggregated rents into his calculation, while a revenue-maximizing seller focuses on the selling price. A priori, releasing information could increase competition at the top such that bidders' rents become smaller. This may affect selling prices a lot, but increase overall efficiency of allocation and thus welfare only marginally. Yet information release could also lead to a further differentiation of the bidders with the highest valuations, thus affecting and increasing bidders' rents and welfare more strongly than the seller's revenue.

Understanding how welfare and revenue incentives relate to each other requires a thorough understanding of the behavior of order statistics. In case of a one-object auction, the first and second order statistics, i.e. the highest and the second highest valuations, and the difference between the two, are crucial. In multi-object auctions, more of the highest order statistics are relevant. If several prizes, like grants or promotions, are "auctioned off" to applicants in order to reward those who exert the highest efforts (bids), efforts of several applicants scratching the top matter. For example, Harvard University selected 2,000 students out of 34,000 applicants for its class of 2018.[1]

The related literature so far focuses on one-object auctions and has mostly imposed conditions on the effects of information release which guarantee that welfare reacts more sensitively towards information than revenue (compare Ganuza and Penalva,

[1] See https://college.harvard.edu/admissions/admissions-statistics.

2010). Yet the opposite conclusion can hold true as well. Bidders' rents may decrease in response to information release due to fiercer competition at the top. This implies that a revenue maximizer has stronger incentives to release information than a welfare maximizer. For instance, this is the case when information release affects bidders with intermediate valuations more strongly than bidders with high valuations.

This paper provides criteria that determine whether information release strengthens or weakens competition in multi-object auctions. The previous literature has typically modeled information release as an increase in the variability of valuations in the sense of the dispersive order (Ganuza and Penalva, 2010).[2] Under the dispersive order, additional information always weakens competition as it increases the differences between all order statistics.

We introduce two new classes of stochastic orders that allow for a more flexible and directed control of the behavior of order statistics, the k- and k-m-dispersion orders. Increased variability in the sense of k-dispersion implies that the k highest order statistics move further apart through information release. Increased variability in the sense of k-m-dispersion implies the same conclusion when the overall number of bidders n is sufficiently large, $n > k + m$. Both orders are weaker than the dispersive order. In particular, information release can either increase or decrease the variability of valuations in k-m-dispersion, implying either a strengthening or a softening of competition. Consequently, a welfare maximizer may have either stronger or weaker incentives to release information than a revenue maximizing seller. k-m-dispersion provides a criterion to decide which of the two is the case.

We apply our theory to auctions in which information release is modeled in terms of information partitions. Bidders do not know their true valuations, yet they know which interval of a distribution contains their valuation. Information release renders these intervals finer. This is a prominent model of information release in economic theory (see Bergemann and Pesendorfer, 2007) that is not tractable with the dispersive order. k-m-dispersion enables us to draw clear conclusions about multi-object

[2]A related literature studies the problem of information acquisition in auctions from the bidder's perspective, e.g., Persico (2000). In there, a bidder compares how different signals affect his valuation estimate. We study the seller's problem in which information release transforms a distribution of unknown valuation estimates into another. Formally, information acquisition is thus a rather different problem that requires different statistical tools such as Blackwell's (1951) sufficiency or Lehmann's (1988) efficiency of signals.

auctions with sufficiently many bidders. Information release decreases bidders' rents if and only if information affects the bidders with the highest valuations.

In a second classical model of information release due to Lewis and Sappington (1994), each bidder's signal equals his valuation with some probability while it is pure noise otherwise. This model has been applied to auctions, e.g., by Ganuza and Penalva (2010) and Shi (2012). k-m-dispersion allows to determine the comparative statics of information release even if signal quality differs for bidders with higher versus lower valuations. While information release often relaxes competition between bidders in this setting, the opposite can happen as well. Specifically, further increases in high signal qualities may foster competition at the top.

Our results directly translate to the problem of understanding the impact of targeted advertising on revenues in auctions, see Hummel and McAfee (2015).[3] Beyond auctions, our techniques also contribute to other fields such as reliability theory and risk management where worst realizations of distributions matter. Differences between order statistics are also crucial in matching markets. Analyzing expected matches between firms and workers, or men and women, requires to control distances between order statistics not only at the top, but also on lower levels of a distribution. Another field of application – beyond the scope of this paper – may be the measurement of inequality, where distances from the poorest (or the richest) to the middle income quantiles of a population may be of specific interest. For example, recent developments in Western countries such as the US suggest that a focus on the distances between the richest 400 families and the middle class could help defining educational goals for the next decades.[4]

Related Literature

This paper is related to several contributions in the literatures on auctions and on stochastic orders.[5] Our auction-theoretic applications generalize results of Ganuza and Penalva (2010) and thus contribute to the literature on information in auctions and mechanism design.[6] Jia, Harstad and Rothkopf (2010) study information release in auctions when bidders know parts of their valuations and the other additive parts can be disclosed. They illustrate that the comparative statistics of bidders' revenues

[3]For a broader picture of the online advertising market, see Athey, Calvano and Gans (2014).
[4]See "America's elite. An hereditary meritocracy", The Economist, 01/24/2015.
[5]For introductions to these two fields, see Krishna (2002) and Shaked and Shanthikumar (2007).
[6]For a survey, see Bergemann and Välimäki (2006).

are intricate and conclude that "no illuminating necessary condition seems possible". This is the problem we address. Stochastic orders, especially the dispersive order, have also been applied to study other questions concerning auctions and related contexts, see, for instance, Johnson and Myatt (2006), Szech (2011), Mares and Swinkels (2014), Kirkegaard (2014), and the references therein.

In the literature on stochastic orders, parts of our analysis build on a result of Li and Shaked (2004) who prove one of the main properties of the k-dispersion order without explicitly introducing this order.[7] We provide new insights on k-dispersion and introduce the generalized k-m-dispersion orders. As the k-dispersion order coincides with the excess wealth order in the case $k = 1$, our results are also related to two contributions from the operations research literature which apply the excess wealth order to auctions, Li (2005) and Xu and Li (2008). Analyzing the case $k > 1$ allows us to address many questions which are not tractable under the excess wealth order. Paul and Gutierrez (2004) provide several results related to ours based on the star order. Yet their results stating that differences of order statistics can be controlled in terms of the star order are incorrect as is shown in Xu and Li (2008).[8]

Outline

Section 2 introduces our model and discusses the scope and limitations of modeling information release in terms of the dispersive order. Section 3 introduces our new stochastic orders and their key properties. We provide some practical sufficient conditions for applications, and show that k-m-dispersion implies a complete ordering on finite distributions. Section 4 presents our main results on information release in multi-object auctions, first in the general case and then in the applications of information partitions and heterogeneous signal quality. Section 5 sketches further economic applications of our methods and presents additional properties of k-dispersion. Section 6 concludes. All proofs are in the appendix.

[7] Compare Proposition 2.

[8] This incorrect result is also cited in Shaked and Shanthikumar (2007) as Theorem 4.B.19.

2 The Setting

2.1 Auction Model with Information Release

We study a symmetric independent private values auction model with information release. Our techniques will allow us to handle one object as well as multi-object auctions. We therefore introduce the broader setting straight away.

A risk-neutral seller auctions off a quantity of q identical objects in a $(q+1)^{th}$ price auction. The $n > q$ bidders are all risk-neutral. Those who submit the q highest bids receive an object and each of them pays the $(q+1)^{th}$ highest bid. Ties are broken with uniform randomness.

Initially, bidders do not know their valuations exactly. Before the auction takes place, the seller decides whether he wants to release information to the bidders. If he opts against information release, the bidders stick to their initial private estimates Y_i of their valuations. The Y_i are nonnegative and independently distributed according to a commonly known cumulative distribution function G with finite mean. If the seller opts for information release, each bidder receives an independent signal that reveals more about his valuation for winning an object. We denote by X_i the updated estimates of valuations. The random variables X_i are again nonnegative, independent and identically distributed with finite mean and we denote their cumulative distribution function by F. F^{-1} and G^{-1} denote the generalized inverse (quantile) functions of F and G.

Throughout we assume that all bidders follow their weakly dominant strategy of bidding their best estimate of their valuation in the auction. Thus, bidder i bids X_i if information is released and Y_i otherwise. We denote by $X_{i:n}$ the i^{th} order statistic, i.e., the i^{th}-largest out of $X_1, \ldots, X_n$, and define $Y_{i:n}$ analogously.[9] Lemma 1 summarizes the main properties of the bidding equilibrium.

Lemma 1 *Set $Z = X$ if information is released and $Z = Y$ if no information is released. The expected selling price in the auction is given by $E[Z_{q+1:n}]$. The seller's expected payoff is given by $q\,E[Z_{q+1:n}]$. Bidders' aggregate rents are given by*

$$\sum_{j=1}^{q} E[Z_{j:n} - Z_{q+1:n}]$$

[9] In particular, we follow the usual notation in auction theory where $X_{1:n}$ denotes the largest order statistic and not the usual statistics notation where it would denote the smallest.

and total welfare amounts to

$$\sum_{j=1}^{q} E[Z_{j:n}].$$

In the following, we call the seller a welfare maximizer if he is interested in maximizing total welfare, and we call him a revenue maximizer if he maximizes his expected payoff.

An alternative interpretation of the model is that F denotes a finer information structure compared to G, and the seller decides whether to release a signal implementing G or F. In the context of information release with Bayesian updating, it is plausible to assume that F and G share the same mean. Our analysis, however, does not rely on this assumption, thus incorporating the possibility of non-Bayesian updating by the bidders. As a final interpretation, the seller could decide between running the auction with bidders from two different populations with respective distributions F versus G.

We do not impose that F and G are continuous. This allows us to provide results for models of information release such as information partitions that would violate a continuity requirement. The additional structures introduced in Ganuza and Penalva (2010) in the one object case – a prior distribution of valuations, a continuous family of signals with associated costs of information provision, and a continuous family of (posterior) distributions of valuations – directly translate to our setting. In particular, while we do not explicitly specify costs of information release, the comparison between F and G should be thought of as one side of a cost-benefit trade-off. While we focus on $(q+1)^{th}$ price auctions, the results can be transferred to more general mechanisms by the revenue equivalence theorem for multi-unit auctions in Engelbrecht-Wiggans (1988) in the case of continuous distributions.

2.2 Information Release and the Dispersive Order

This section illustrates how measures of dispersion allow to study the effects of information release in auctions. We provide an overview of existing results and point out their limitations by an example.

Intuitively, providing more information to bidders should increase the variability in their estimated valuations. The posterior distribution F should thus be more variable (or "dispersed") than the prior G. In their analysis of information release, Ganuza and Penalva (2010) study two notions of dispersion, an ordering between F and G in the convex order, and an ordering of F and G in the dispersive order. These are defined as follows.[10]

Definition 1

(i) F is more variable than G in the convex order, $F \succeq_{conv} G$,[11] if $E[X_1] = E[Y_1]$ and

$$E[(X_1 - t)^+] \;\geq\; E[(Y_1 - t)^+] \;\; \textit{for all } t \in \mathbb{R}$$

where $(\cdot)^+$ denotes the positive part.

(ii) F is more variable than G in the dispersive order, $F \succeq_{disp} G$, if

$$F^{-1}(p) - F^{-1}(q) \geq G^{-1}(p) - G^{-1}(q) \quad \textit{for all } \; 0 < q < p < 1. \tag{1}$$

An ordering in the convex order is a weak requirement closely related to second-order stochastic dominance. It is satisfied in many models of information release. Under the assumption that $F \succeq_{conv} G$, Ganuza and Penalva show that releasing information increases expected welfare and, with sufficiently many bidders, the expected revenue in the auction.[12] Both results follow from the intuition that increasing the variability of valuations tends to increase the highest valuations.

In order to control differences between overall welfare and seller's revenues, stronger orderings need to be imposed. Ganuza and Penalva rely on the dispersive order. F dominates G in the dispersive order if all pairs of quantiles lie further apart under F than under G. As we will see below, this is a rather rigid requirement which is violated in many models of information release. The next lemma summarizes their results on information release in auctions under the assumption that $F \succeq_{disp} G$.[13]

Lemma 2 *Assume $F \succeq_{disp} G$ and $q = 1$.*

[10] For background on these two orders, see Chapters 3.A and 3.B of Shaked and Shanthikumar (2007). Our definitions follow their Theorem 3.A.1 and Formula 3.B.1.

[11] For our purposes, it proves to be more convenient to formulate stochastic orders on the level of distribution functions and not on the level of random variables as is done, e.g., in Shaked and Shanthikumar (2007).

[12] These results are their Theorems 3 and 5. For a generalization to the q object case, see Roesler (2015).

[13] The four parts of Lemma 2 correspond to Proposition 6, Theorem 7, Theorem 4 and Theorem 6 of Ganuza and Penalva (2010).

(i) Bidders' aggregate rents increase when information is released,

$$E[X_{1:n} - X_{2:n}] \geq E[Y_{1:n} - Y_{2:n}].$$

(ii) A welfare maximizing seller has a stronger incentive to release information than a revenue maximizing seller,

$$E[X_{1:n} - Y_{1:n}] \geq E[X_{2:n} - Y_{2:n}].$$

(iii) The expected welfare generated by the auction increases more strongly when the number of bidders increases under information release than when no information is released,

$$E[X_{1:n} - X_{1:n-1}] \geq E[Y_{1:n} - Y_{1:n-1}].$$

(iv) The seller's expected payoff increases more strongly when the number of bidders increases under information release than when no information is released,

$$E[X_{2:n} - X_{2:n-1}] \geq E[Y_{2:n} - Y_{2:n-1}].$$

All four results rely on comparisons of differences of order statistics, so-called spacings. Technically, they stem from the following fact about the dispersive order.[14]

Lemma 3 *Let $F \succeq_{disp} G$. Then for all $i < n$*

$$E[X_{i:n} - X_{i+1:n}] \geq E[Y_{i:n} - Y_{i+1:n}]$$

and

$$E[X_{i:n} - X_{i:n-1}] \geq E[Y_{i:n} - Y_{i:n-1}].$$

In the remainder of this section, we illustrate a setting which does not fall under Lemma 2 and which leads to the opposite economic implications.

Example 1

Assume that bidders' true valuations are distributed uniformly on $[0, 1]$. Bidders do not know their true valuations. They only know whether their valuation is below $2/3$ or not. By releasing information, the seller can furnish bidders with the additional

[14] The first claim of Lemma 3 follows from Theorem 3.B.31 of Shaked and Shanthikumar (2007). The second claim follows from the first and formula (9) below.

information whether their valuations lie below or above $1/3$*. Consequently, the a priori distribution* G *puts a mass of* $2/3$ *on the value* $1/3$ *and the remaining mass on* $5/6$.[15] *The a posteriori distribution* F *is a uniform distribution on* $1/6$, $1/2$ *and* $5/6$*. Notice first that* F *and* G *are not comparable in the dispersive order. When moving from* G *to* F *the lowest third of probability mass moves downwards from* $1/3$ *to* $1/6$ *while the middle third moves upwards from* $1/3$ *to* $1/2$*. The upper quantiles do not react to the information release. Therefore, the lower two-thirds of probability mass are indeed more dispersed under* F *than under* G*. Yet the upper two-thirds lie more closely together.*

When working with information partitions, information release will always lead to such ambiguous effects and thus preclude a direct application of the dispersive order. In Section 4.2 below, we discuss in more detail how this example relates to information partitions in general.

As Lemma 2 is not applicable in our example, we compare welfare and seller's revenues by a direct calculation,

$$E[X_{1:n} - X_{2:n}] = \frac{1}{9} n \left(\frac{2}{3}\right)^{n-1} \left(1 + \left(\frac{1}{2}\right)^{n-2}\right) \quad \textit{and} \quad E[Y_{1:n} - Y_{2:n}] = \frac{1}{6} n \left(\frac{2}{3}\right)^{n-1}.$$

For $n = 2$*, we obtain results similar to parts (i) and (ii) of Lemma 2. For* $n = 3$*, welfare and seller's revenues react equally strongly. With four or more bidders, the results are reversed. Bidders' aggregate rents decrease when information is released. Thus a revenue maximizing seller has a stronger incentive to release information than a welfare maximizing one.*[16]

In our example, information affects bidders with intermediate valuations more strongly than bidders with high valuations. This renders the auction more competitive. In particular, information release does not increase the differences between high order statistics. If we look at restrictions of F and G to sufficiently high quantiles, we see that, in a sense, information release reduces dispersion.

Definition 2 *For* $p \in (0, 1)$ *define the restriction of* F *to its quantiles higher than*

[15] For a more detailed introduction of this model, see Section 4.2.

[16] As we will see in greater generality in Section 4.2, parts (iii) and (iv) of the lemma are also reversed with sufficiently many bidders.

p as the cumulative distribution function

$$F_{>p}(x) = \begin{cases} \frac{F(x)-p}{1-p} & x \geq F^{-1}(p) \\ 0 & x < F^{-1}(p) \end{cases}$$

and define $G_{>p}(x)$ analogously.[17]

Consider the distributions $F_{>1/3}$ and $G_{>1/3}$. $F_{>1/3}$ is the uniform distribution on $\{1/2, 5/6\}$ while $G_{>1/3}$ is the uniform distribution on $\{1/3, 5/6\}$. Unlike F and G themselves, these restrictions can be compared in the dispersive order. Yet it is the distribution without information release which is more dispersed, $G_{>\frac{1}{3}} \succeq_{disp} F_{>\frac{1}{3}}$. Since higher quantiles dominate the behavior of high order statistics with sufficiently many bidders, this observation explains the reversal of Lemma 2. Indeed, we will see in Proposition 6 and Theorem 1 that a dispersive ordering between F and G above some quantile is essentially a sufficient condition for whether Lemma 2 holds or whether it is reversed.

3 Dispersion Criteria for Order Statistics

As seen in Lemma 3, the dispersive order implies a control over *all* spacings of order statistics while the outcomes of auctions depend only on the highest few. This motivates the k-dispersion order, which is specifically designed to control spacings of the k highest order statistics. This family of stochastic orders focuses on the properties of a distribution that are crucial for an auction's outcome, and avoids imposing more restrictions than needed.

Even in situations in which a clear monotonicity behavior of high spacings does not exist in general, it may emerge as soon as sufficiently many bidders take part in an auction. This is demonstrated in Example 1, and motivates us to introduce the family of k-m-dispersion order. These stochastic orders allow to control the k highest order statistics in auctions with more than $k+m$ bidders. We then provide sufficient conditions for k- and k-m-dispersion that are easy to verify in applications. Finally, we show the following completeness result: Any pair of finite distributions is comparable in k-m-dispersion when the parameter m is chosen sufficiently large.

[17] Notice that the definition is such that if F has an atom on $F^{-1}(p)$, i.e., $F(F^{-1}(p)) = s > p$ then $F_{>p}(x)$ has an atom of size $(s-p)/(1-p)$ on $F^{-1}(p)$.

3.1 k-Dispersion

This section introduces the family of k-dispersion orders, compares them with other stochastic orders, and develops their implications.

Definition 3 (k-Dispersion) *For an integer $k \geq 1$, F is more dispersed than G in the k-dispersion order, $F \succeq_k G$, if*

$$\int_p^1 (1-u)^k \, dF^{-1}(u) \geq \int_p^1 (1-u)^k \, dG^{-1}(u) \tag{2}$$

for all $p \in (0,1)$.

While our proofs are based on (2), the following alternative formulations of this condition may be easier to interpret. We can write (2) as[18]

$$\int_{F^{-1}(p)}^{\infty} (1-F(x))^k dx \geq \int_{G^{-1}(p)}^{\infty} (1-G(x))^k dx,$$

and as

$$E[(X_{k:k} - F^{-1}(p))^+] \geq E[(Y_{k:k} - G^{-1}(p))^+]. \tag{3}$$

From (3), we see that F is more k-dispersed than G if upwards deviations of the smallest out of k draws are greater in expectation under F than under G. Compared to the definition of the convex order in (1), there are two differences. First, the reference levels for deviations are the p-quantiles of the two distributions rather than the same fixed reference level on both sides of the inequality. Intuitively, since order statistics are connected to quantiles, this is the reason why k-dispersion allows to draw stronger conclusions about spacings of order statistics than the convex order. The probability that $X_{k:n}$ lies above the p-quantile is the same for all distributions. This makes quantiles a good starting point for comparing order statistics. Second, for $k > 1$, we directly impose a condition on $X_{k:k}$ rather than on $X_{1:1}$. The parameter k thus allows us to gradually adjust the strength of the dispersion criterion to the level that is needed. In Propositions 2 and 3 below, we show that k-dispersion allows to control spacings of the k highest order statistics for all n. In Section 5, we demonstrate its applicability to order statistics that lie further apart as well as to normalized spacings.

[18] These equivalences are implicit in the proof of Proposition 3.4 of Li and Shaked (2004), see also Section 2 of Broniatowski and Decurninge (2015) for the relevant integral substitution formulas. We only work with formulation (2) in the following and thus omit the calculations here.

Throughout the paper, we mostly apply k-dispersion by relying on variations of the following argument. Condition (2) implies that for any increasing function h

$$\int_0^1 h(u)(1-u)^k \, dF^{-1}(u) \geq \int_0^1 h(u)(1-u)^k \, dG^{-1}(u). \tag{4}$$

For non-negative random variables, spacings of order statistics can be written as[19]

$$E[X_{k:n} - X_{k+1:n}] = \binom{n}{k} \int_0^1 u^{n-k}(1-u)^k dF^{-1}(u), \tag{5}$$

where $k < n$. Thus, choosing $h(u) = u^{n-k}$ in (4) shows that k-dispersion implies a ranking of spacings.

Main Characteristics of k-Dispersion

We next explore the main characteristics of k-dispersion and compare it to the stochastic orders typically applied in economics, the dispersive order and the convex order. $\succeq_k$ is a stochastic (partial) order in that it is transitive:[20] For three distribution functions F, G, and H, $F \succeq_k G$ and $G \succeq_k H$ imply $F \succeq_k H$. While the 1-dispersion order coincides with the excess wealth order,[21] the k-dispersion orders for $k > 1$ appear to be novel.[22] Like the excess wealth order, all k-dispersion orders are location independent, i.e., $F \succeq_k G$ remains fulfilled if either of the two distributions is shifted by a constant.

Proposition 1

(i) For all $k \geq 1$, if $F \succeq_{disp} G$ then $F \succeq_k G$.

(ii) For all $k \geq 1$, if $F \succeq_{k+1} G$ then $F \succeq_k G$.

(iii) For all $k \geq 1$, if $E[X_1] = E[Y_1]$ and $F \succeq_k G$ then $F \succeq_{conv} G$.

Thus, the dispersive order is stronger (and less broadly applicable) than all k-dispersion orders. For instance, it is a necessary condition for the dispersive order that F^{-1} and G^{-1} cross only once. k-dispersion does not rely on such a single-crossing condition.

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[19] For a derivation, see, e.g., Kadane (1971).

[20] This separates k-dispersion from some single-crossing criteria for dispersion such as the rotation criterion of Johnson and Myatt (2006).

[21] See Shaked and Shanthikumar (2007) for background on the excess wealth order.

[22] The definition is motivated by an observation of Li and Shaked (2004), see Proposition 2 below.

Within the family of k-dispersion orders, $(k+1)$-dispersion implies k-dispersion. The convex order can generally not be compared to k-dispersion and the dispersive order as it is not location independent: $F \succeq_{conv} G$ can only hold if F and G have the same mean. Under the assumption that F and G share the same mean, the convex order is implied by each of the other orderings. Yet the convex order itself is not strong enough to control spacings of order statistics.

k-Dispersion and Order Statistics

Proposition 2 demonstrates the suitability of k-dispersion for controlling spacings of high order statistics. The result combines Proposition 1 (ii) with Proposition 3.4 of Li and Shaked (2004).

Proposition 2 *If $F \succeq_k G$ for some $k < n$ then for all $i \leq k$*

$$E[X_{i:n} - X_{i+1:n}] \geq E[Y_{i:n} - Y_{i+1:n}].$$

Next, we extend this result to the other class of spacings of order statistics where we vary n while keeping i fixed. The key observation is that the two types of spacings differ only by a combinatorial factor which is not distribution-dependent.

Proposition 3 *If $F \succeq_k G$ for some $k < n$ then for all $i \leq k$*

$$E[X_{i:n} - X_{i:n-1}] \geq E[Y_{i:n} - Y_{i:n-1}].$$

In the context of auctions, Proposition 3 enables us to study how welfare and revenue react to changes in the number of bidders.

Finally, a word is in order on comparisons of expected order statistics. k-dispersion is designed to be location independent to facilitate comparisons of differences. In order to obtain results comparing $E[X_{k:n}]$ and $E[Y_{k:n}]$, the two distributions need to be anchored in fixed locations. For instance, when F and G share the same mean, comparison results of this type can be derived from the fact that k-dispersion implies the convex order.

3.2 k-m-Dispersion

In Example 1, monotonicity of spacings sets in only with sufficiently many bidders. While k-dispersion is weaker than the dispersive order, it cannot apply in such a situation. Building on k-dispersion, we therefore introduce the weaker concept of k-m-dispersion. This implies that Propositions 2 and 3 hold in richer settings if the number of bidders is sufficiently large, namely $n > k + m$.

Definition 4 (k-m-Dispersion) *For integers $k \geq 1$ and $m \geq 0$, F is more dispersed than G in the k-m-dispersion order, $F \succeq_{k,m} G$, if*

$$\int_p^1 u^m (1-u)^k dF^{-1}(u) \geq \int_p^1 u^m (1-u)^k \, dG^{-1}(u) \tag{6}$$

for all $p \in (0,1)$.

All k-m-dispersion orders are location-independent and transitive. k-0-dispersion coincides with k-dispersion. In general, compared to k-dispersion, the increasing function u^m in the integrand shifts attention into the right tail of the distribution. With many bidders, the behavior at this tail is crucial for an auction's outcomes. Analogously to the alternative condition (3) for k-dispersion, we can write condition (6) as

$$\begin{aligned} & E[(X_{k:k+m} - F^{-1}(p))^+] - E[(X_{k+1:k+m} - F^{-1}(p))^+] \\ \geq \;& E[(Y_{k:k+m} - G^{-1}(p))^+] - E[(Y_{k+1:k+m} - G^{-1}(p))^+]. \end{aligned} \tag{7}$$

Thus, k-m-dispersion postulates that F is more dispersed than G in the following sense. Exchanging the $(k+1)^{th}$-largest for the k^{th}-largest out of $k+m$ draws increases deviations above a given quantile more strongly under F than under G. This exchange of order statistics is one way of moving further into the right tail of the distribution.[23]

Therefore, k-m-dispersion should hold when the right tail of F is more spread out than the right tail of G. This is the case even when a global comparison of the dispersion of the two distributions is not instructive, as in Example 1. Proposition 4 summarizes the central properties of k-m-dispersion regarding spacings of order statistics. The proposition generalizes Propositions 2 and 3.

[23] Li and Shaked (2004) point out that (3) is equivalent to postulating that the distributions of $X_{k:k}$ and $Y_{k:k}$ are ordered in the excess wealth order $\succeq_1$. k-m-dispersion cannot be reduced to the excess wealth order in such a way.

Proposition 4

(i) If $F \succeq_{k,m} G$ *for some* k *and* m *with* $k+m<n$ *then for all* $i \le k$

$$E[X_{i:n} - X_{i+1:n}] \ge E[Y_{i:n} - Y_{i+1:n}].$$

(ii) If $F \succeq_{k,m} G$ *for some* k *and* m *with* $k+m<n$ *then for all* $i \le k$

$$E[X_{i:n} - X_{i:n-1}] \ge E[Y_{i:n} - Y_{i:n-1}].$$

To put the k-m-dispersion orders into context we add the following result in the spirit of Proposition 1.

Proposition 5

(i) For all $k \ge 1$ *and for all* $m \ge 0$*, if* $F \succeq_{k,m} G$ *then* $F \succeq_{k,m+1} G$*.*

(ii) For all $k \ge 1$ *and for all* $m \ge 0$*, if* $F \succeq_{k+1,m} G$ *then* $F \succeq_{k,m} G$*.*

(iii) If $k, m \ge 1$ *and* $E[X_1] = E[Y_1]$ *then* $F \succeq_{k,m} G \not\Rightarrow F \succeq_{conv} G$ *and* $F \succeq_{conv} G \not\Rightarrow F \succeq_{k,m} G$*.*

Accordingly, increasing m renders the k-m-dispersion order less rigid. All k-m-dispersion orders are weaker than the k-dispersion order and, consequently, the dispersive order. Unlike k-dispersion, k-m-dispersion is not comparable to the convex order if F and G share the same mean. Indeed, in the application to information partitions in Section 4.2, we find that $F \succeq_{conv} G$ is always satisfied while either $F \succeq_{k,m} G$ or $G \succeq_{k,m} F$ holds. In an auction setting, this implies that welfare always increases while bidders' revenues either increase or decrease through information release. The dispersive order is not applicable in these examples since it cannot handle information partitions.

3.3 Sufficient Conditions for k-m-Dispersion

We now introduce some more explicit conditions for k-m-dispersion that allow to derive quantitative results without actually computing the integrals in (6). The starting point is a dispersive ordering between restrictions of F and G to high quantiles as introduced in Definition 2.

Proposition 6 *Suppose there exists $p \in (0,1)$ such that $F_{>p} \succeq_{disp} G_{>p}$ and there exist $q_1, q_2 \in (p,1)$, $q_2 > q_1$, such that the constant*

$$d_+ = \left(F^{-1}(q_2) - F^{-1}(q_1)\right) - \left(G^{-1}(q_2) - G^{-1}(q_1)\right)$$

is positive. Define also $d_- = G^{-1}(p^+) - G^{-1}(0^+)$. Then, for any k and $m \geq \hat{m}(k)$ given by

$$\hat{m}(k) = \left\lceil \frac{k \cdot \log\left(\frac{1}{1-q_2}\right) + \log\left(\frac{d_-}{d_+}\right)}{\log(q_1) - \log(p)} \right\rceil,$$

we have $F \succeq_{k,m} G$. Here, $\lceil \cdot \rceil$ denotes rounding to the next non-negative largest integer.

In the proposition, the condition $d_+ > 0$ ensures that the comparison in the dispersive order holds, in a sense, strictly. d_+ serves as a quantitative measure of the additional dispersion of F versus G above the p-quantile. d_+ is compared to the constant d_- which quantifies how spread out G can maximally be outside this region. d_- is the difference between the lower ends of the supports of G and $G_{>p}$.[24] The thresholds $\hat{m}(k)$ grow linearly in k, and so do the implied thresholds $\hat{n}(k) = k + \hat{m}(k)$ on n. The intuition for this linear relationship is that the typical location of $X_{k:n}$ among the quantiles of F is linked to the ratio k/n.

When applying the proposition, in order to obtain a small threshold $\hat{m}$, q_1 and q_2 should be chosen at some distance from each other and away from the boundaries of the interval $(p,1)$. This ensures that d_+ is sufficiently large, and that the region between the q_1- and q_2-quantiles becomes more important than the region below.[25] Finally, notice that when $\hat{m}(k)$ is zero, Proposition 6 implies k-dispersion.

Example 1: Explicit Thresholds

In Example 1, we wish to conclude $G \succeq_{k,m} F$ from $G_{>1/3} \succeq_{disp} F_{>1/3}$. We thus need to apply Proposition 6 with the roles of F and G exchanged. We choose $p = 1/3$, $q_1 = 2/3$ and $q_2 = 3/4$ which gives $F^{-1}(q_2) = G^{-1}(q_2) = 5/6$, $F^{-1}(q_1) = 1/2$, $G^{-1}(q_1) = 1/3$ and thus $d_+ = 1/6$. $F^{-1}(0^+) = 1/6$ and $F^{-1}(p^+) = 1/2$ imply

[24] The values 0^+ and p^+ in the definition of d_- are needed to properly define the lower ends of the supports. For any distribution function F on $\mathbb{R}$, $F^{-1}(0^+) = \inf\{x | F(x) > 0\}$ is the lower end of the support (which is non-negative by our assumptions) while $F^{-1}(0) = \inf\{x | F(x) \geq 0\} = -\infty$.

[25] In particular, $q_2 < 1$ guarantees a uniform control over the contribution of the q_1-q_2-region, because the very highest quantiles contribute little to intermediate order statistics.

k	1	2	3	4	5	10	15	20
$\hat{n}(k)$	4	7	10	13	16	31	46	61
$n^*(k)$	3	5	7	9	11	21	31	41

Table 1: Estimated thresholds $\hat{n}$ and true minimal thresholds n^* for n to ensure $E[Y_{k:n} - Y_{k+1:n}] > E[X_{k:n} - X_{k+1:n}]$ in Example 1 for different k.

$d_- = 1/3$. *It follows that* $G \succeq_{k,m} F$, *provided that*

$$m \geq \hat{m}(k) = 2 \cdot k + 1.$$

Table 1 compares the estimated thresholds $\hat{n}(k) = k + \hat{m}(k)$ *to the true thresholds* $n^*(k)$.[26] *The inequality* $E[Y_{k:n} - Y_{k+1:n}] > E[X_{k:n} - X_{k+1:n}]$ *holds if and only if* $n > n^*(k)$. *Proposition 6 proves the inequality for* $n > \hat{n}(k)$. *The true thresholds* n^* *all satisfy* $n^*(k) = 2 \cdot k + 1$. *The estimated thresholds* $\hat{n}(k)$ *reflect this linear growth behavior. We further see that the estimated slope of* 3 *has a reasonable magnitude.*

Counting downwards from the upper bound $\hat{n}(k)$, one can determine the optimum $n^*(k)$ by direct calculation. Without an upper bound, finding $n^*(k)$ would not be possible. Counting upwards from k would be problematic as the ordering of $E[Y_{k:n} - Y_{k+1:n}]$ and $E[X_{k:n} - X_{k+1:n}]$ might change arbitrarily often.

For discrete distributions, the condition $F_{>p} \succeq_{disp} G_{>p}$ can be verified by comparing the upper tails of the distributions. The final result of this section provides a similar criterion for continuous distributions on bounded supports. $F \succeq_{k,m} G$ holds for sufficiently large m if the density of F is smaller than the density of G near the upper ends of the respective supports.

Corollary 1 *Suppose* F *and* G *are continuous with bounded supports* $[a_F, b_F]$ *and* $[a_G, b_G]$ *and possess continuous density functions* f *and* g. *If there exists* $\delta > 0$ *such that* $0 < f(x) < g(y)$ *for all* $x \in [b_F - \delta, b_F]$ *and* $y \in [b_G - \delta, b_G]$ *then for any* k *there exists* m *such that* $F \succeq_{k,m} G$.

Thus it is the thickness of the density in the upper tail, not its location, that ultimately dictates the behavior of spacings of high order statistics. In contrast, the ordering of $E[X_{k:n}]$ and $E[Y_{k:n}]$ for large n depends on the location of the upper tail, i.e., on a comparison of the upper boundaries of the supports $F^{-1}(1)$ and $G^{-1}(1)$.

[26] n^* was computed numerically.

3.4 A Completeness Result

This section shows that any pair of finite distributions can be compared in k-m-dispersion, i.e., there always exists a value of m such that k-m-dispersion is applicable.[27] In this sense, we obtain a complete ordering.

We first show that for any pair of finite distributions that are not identical up to horizontal shifts, there exists some p such that $F_{>p}$ and $G_{>p}$ are ordered strictly in the dispersive order. Here, the strict ordering $F_{>p} \succ_{disp} G_{>p}$ means that $F_{>p} \succeq_{disp} G_{>p}$ holds while $G_{>p} \succeq_{disp} F_{>p}$ is violated. Restricting attention to distributions that are not horizontal shifts of each other is without loss of generality. If the distributions are horizontal shifts, they are equivalent under $\succeq_{disp}$ and $\succeq_{k,m}$.

Proposition 7 *Suppose the distributions F and G are finite and they are not horizontal shifts of each other. Then there exists a $p \in (0,1)$ such that $F_{>p} \succ_{disp} G_{>p}$ or $G_{>p} \succ_{disp} F_{>p}$.*

Now define the stochastic order $\succeq_{disp*}$ as follows. $F \succeq_{disp*} G$ holds whenever there exists a $p \in (0,1)$ such that $F_{>p} \succeq_{disp} G_{>p}$. Proposition 7 implies that $\succeq_{disp*}$ is a complete order on finite distributions. The next corollary shows that this completeness is inherited by the orders $\succeq_{k,*}$ defined analogously. $F \succeq_{k,*} G$ holds whenever there exists an m such that $F \succeq_{k,m} G$.

Corollary 2 *For any two finite distributions F and G that are not horizontal shifts of each other, the following three claims are equivalent.*

(i) There exists p such that $F_{>p} \succ_{disp} G_{>p}$.

(ii) For all k, there exists an m such that $F \succeq_{k,m} G$.

(iii) There exist k and m such that $F \succeq_{k,m} G$.

Letting m converge to infinity leads to the complete order $\succeq_{k,*}$, which is a completion of the dispersive order. As m increases, more pairs of distributions become comparable in k-m-dispersion. This will help us to derive results about auctions with at least $k+m$ bidders in Section 4.

The proof of Proposition 7 is constructive, i.e., it implies an explicit algorithm for deciding which of the two distributions F versus G is more dispersed in k-m

[27] We thus assume that the supports of F and G have finite cardinality. We conjecture that for general distributions F and G counterexamples may be constructed by considering densities which intersect infinitely often in the tail.

dispersion for sufficiently large m. We present two results in this vein in the context of information partitions in Section 4.2.

4 Information Release in Multi-Object Auctions

4.1 The General Case

This section applies k- and k-m-dispersion to information release in auctions in which q identical objects are for sale. Each bidder is in need of one of these objects, as e.g. in contests for promotions or admission to a university.

Theorem 1 generalizes Lemma 2. It provides conditions for welfare reacting more strongly to information than seller's revenues, as well as conditions for the opposite situation. Furthermore, it covers the cases in which sufficiently many bidders need to take part in order to arrive at clear-cut results.

Theorem 1

(i) If $F \succeq_{q,m} G$ and $n > q + m$, then bidders' aggregate rents increase when information is released.

(ii) If $F \succeq_{q,m} G$ and $n > q + m$, then a welfare maximizing seller has a stronger incentive to release information than a revenue maximizing seller.

(iii) If $F \succeq_{q,m} G$ and $n > q + m$, then the welfare generated by the auction increases more strongly when the number of bidders increases under information release than when no information is released.

(iv) If $F \succeq_{q+1,m} G$ and $n > q + 1 + m$, then the expected selling price and the seller's payoff increase more strongly when the number of bidders increases under information release than when no information is released.

(v) The conclusions of (i-iii) are reversed if $G \succeq_{q,m} F$ and $n > q+m$. The conclusion of (iv) is reversed if $G \succeq_{q+1,m} F$ and $n > q + 1 + m$.

Thus, in the setting $q = 1$ and $m = 0$ of Ganuza and Penalva (2010), the excess wealth order is sufficient for (i) to (iii).[28] Stronger orders with $k = 2$ are needed for (iv), i.e., for understanding the interplay of information release and number of bidders with regard to revenue. We also need stronger dispersion criteria when the

[28]Compare Li (2005) for similar results relying on the excess wealth order.

number of objects, q, increases. Finally, part (v) of the result takes into account that the effects of information release may run in both directions and provides criteria for both cases. An immediate consequence of (ii) is that if information release is costly then for intermediate cost levels a welfare maximizer releases information while a revenue maximizer does not.

4.2 Information Partitions

When information release takes the form of increasingly finer information partitions, Theorem 1 yields a complete characterization of information release with sufficiently many bidders. If information release increases the highest valuation estimate, the requirements of claims (i) to (iv) of the theorem are fulfilled. If the highest valuation estimate is unaffected by information release, the four claims are reversed.

Assume that bidders' true valuations are distributed according to a continuous distribution function H with a strictly positive density h on an interval $[a, b]$ with $a \geq 0$ and $a < b \leq \infty$. Denote by $(\beta_i)_i$ an strictly increasing subsequence of (a, b) with $B > 0$ elements. Thus, β_1 and β_B are the lowest and highest values in the sequence. Without information release, bidders only know for each of the values β_i whether their valuations lie above or below. Accordingly, the distribution G of valuation estimates assigns probability

$$H(\beta_i) - H(\beta_{i-1}) \quad \text{to the estimate} \quad \frac{\int_{\beta_{i-1}}^{\beta_i} xh(x)dx}{H(\beta_i) - H(\beta_{i-1})} \tag{8}$$

with the obvious modifications for β_1 and β_B.

Information release is modeled such that the seller increases the number of values for which bidders know whether their valuation lies above or below. The sequence $(\beta_i)_i$ is thus replaced by another strictly increasing sequence $(\alpha_i)_i$ with $A > B$ elements. $(\beta_i)_i$ is a subsequence of $(\alpha_i)_i$. The distribution F of posterior valuation estimates is derived from $(\alpha_i)_i$ analogously to (8).

Proposition 8 shows that for any k, F and G are always comparable in the k-m-dispersion order for sufficiently large m.

Proposition 8

(i) If $\alpha_A = \beta_B$, then for any k there exists an m such that $G \succeq_{k,m} F$.

(ii) If $\alpha_A > \beta_B$, then for any k there exists an m such that $F \succeq_{k,m} G$.

Whether F or G is more dispersed thus depends on whether information release affects the highest valuation estimates or not. If $\alpha_A = \beta_B$, the bidders with the highest valuation estimates are not affected by information release. The auction thus becomes more competitive such that the reverses of claims (i-iv) of Theorem 1 hold with sufficiently many bidders. If $\alpha_A > \beta_B$, information release further differentiates the valuation estimates of the highest valuation bidders. Consequently, the auction becomes less competitive and the four claims of Theorem 1 hold with sufficiently many bidders.

Let us now turn to the more general case in which F and G result from two different information partitions. We thus drop the assumption that F is a refinement of G. We know from Section 3.4 that any pair of (finite) partitions is comparable in k-m-dispersion for sufficiently large m. The following corollary shows that when one partition differentiates more strongly at the very top, the resulting distribution is the more dispersed one.

Corollary 3 *Let $(\alpha_i)_i$ and $(\beta_i)_i$ be any pair of strictly increasing subsequences of (a, b) and denote by F and G the corresponding distributions of valuation estimates.*

(i) If $\alpha_A > \beta_B$, then for any k there exists an m such that $F \succeq_{k,m} G$.

(ii) If $\alpha_A < \beta_B$, then for any k there exists an m such that $G \succeq_{k,m} F$.

For example, suppose H is the uniform distribution on $[0, 1]$ and F and G are generated from the partitions $\alpha = (1/3, 2/3)$ and $\beta = (1/2)$. In this case, F is more differentiated at the top than G, and thus $F \succeq_{k,m} G$ for sufficiently large m. Note that F and G are not comparable in the convex order or in k-dispersion with $m = 0$.[29]

4.3 Heterogeneous Signal Quality

The quality of a signal may depend on the bidder's type. In this section, we provide a model that allows to study such a heterogeneity in signal quality in a multi-object

[29] The result about the convex order can be seen as follows. When a bidder knows that his valuation is below (or above) 1/2, there is no additional piece of information that leads him to believe that it can lie anywhere between 1/3 and 2/3. This is a general feature of "overlapping" information partitions. The result about k-dispersion follows since k-dispersion would imply the convex order.

auction context. We start with the following classical set-up. Bidders receive a noisy signal which is identical to their valuation with some probability and pure noise otherwise. With homogeneous signal quality, this is the truth-or-noise model introduced in Lewis and Sappington (1994) and applied, e.g., by Johnson and Myatt (2006), Ganuza and Penalva (2010), and Shi (2012). We study a variation of this model which captures heterogeneity in signal qualities. The probability of the signal being correct is different for bidders with high versus low valuations. Possible interpretations include information which is more vital to bidders with low valuations than to bidders with high ones (or vice versa), or, more generally, information which is more precise in some respects than in others.

Bidders' true valuations Z_i are independent and uniformly distributed on $[0,1]$. Each agent receives an independent signal S_i which is either equal to Z_i or equal to U_i where U_i is independent of Z_i and also uniformly distributed on $[0,1]$. There are numbers $\theta, p_L, p_H \in (0,1)$ such that the probability of $S_i = Z_i$ is p_L for $Z_i \leq \theta$ and p_H for $Z_i > \theta$. Signal quality thus depends on whether the true valuation is above or below θ. We denote by G the distribution of valuation estimates which follows from this specification of θ, p_L and p_H.

In this model, releasing more information corresponds to improvements in the quality of the signals. It can thus take three basic forms, an increase in p_H, an increase in p_L or a shift of θ such that more agents have the higher signal quality. In the following, we refer to these three possibilities as an H-increase in information, an L-increase in information, and a T-increase in information.[30] We denote by F the distribution of valuation estimates which arises from either of these increases in the amount, or quality, of information. In particular, we say that F differs from G, e.g., through an H-increase in information if the two distributions are based on the same values of p_L and θ but if F has a higher value of p_H than G.

In order to study the impact of information release, we need to establish what the distributions G and F look like. The probability q_L of observing a signal below θ is given by

$$q_L = P(S_i \leq \theta) = \theta p_L + \theta^2(1-p_L) + \theta(1-\theta)(1-p_H) =: p_1 + p_2 + p_3,$$

[30]For the case of a T-increase, more information is released if $p_H > p_L$ and θ decreases, or if $p_H < p_L$ and θ increases. When $p_H = p_L$, changes in θ have no effect. We thus implicitly assume $p_L \neq p_H$ when speaking of a T-increase in information.

where the three summands p_j correspond to the cases where $S_i = Z_i \leq \theta$, where $S_i, Z_i \leq \theta$ but $S_i \neq Z_i$, and where $S_i \leq \theta$ but $Z_i > \theta$. Analogously, we have

$$q_H = P(S_i > \theta) = (1-\theta)p_H + (1-\theta)^2(1-p_H) + \theta(1-\theta)(1-p_L) =: p_4 + p_5 + p_6.$$

The valuation estimate (and bid) of a bidder who received the signal realization $s \leq \theta$ is thus given by

$$e_L(s) = \frac{1}{q_L}\left(s\, p_1 + \frac{\theta}{2}\, p_2 + \frac{1+\theta}{2}\, p_3\right)$$

where the pre-factors of p_2 and p_3 are the means of uniform distributions on $[0,\theta]$ and $(\theta, 1]$. Similarly, an agent who received $s > \theta$ has the estimate

$$e_H(s) = \frac{1}{q_H}\left(s\, p_4 + \frac{1+\theta}{2}\, p_5 + \frac{\theta}{2}\, p_6\right).$$

Denote by $U(\cdot \mid I\,)$ the density of a uniform distribution on the interval I. Since signals remain uniformly distributed conditional on lying above or below θ, the distribution of valuation estimates G is a mixture of two uniform distributions and its density g is given by $g(y) = q_L\, U(y \mid I_L\,) + q_H\, U(y \mid I_H\,)$ where

$$I_L = [e_L(0), e_L(\theta)] \quad \text{and} \quad I_H = (e_H(\theta), e_H(1)].$$

In this model, an increase in the amount of information does not necessarily imply a higher dispersion in the sense of the dispersive order.[31] Moreover, higher values of the signal realization do not necessarily imply higher valuation estimates. Such a lack of monotonicity can occur if θ is sufficiently large so that $Z_i < \theta$ can still correspond to a rather high valuation, and if signal realizations below θ are more reliable than those above, $p_L \gg p_H$. For the auction, we need to determine whether the overall highest bids come from bidders with the highest possible signals (near 1), or from bidders with signals near θ. This motivates the following definition of monotonicity at the top (MT).

Definition 5 *The tuple (p_L, p_H, θ) satisfies monotonicity at the top* (MT) *if* $e_H(1) >$

[31]For instance, for $p_L = \theta = 0.25$ and $p_H = 0.1$, there is a gap between the two parts of the support I_L and I_H. Improving signal quality by increasing p_H to 0.25 closes this gap, $e_L(\theta) = e_H(\theta)$, so that some quantiles lie more closely together than before, thus ruling out an ordering in the dispersive order.

$e_L(\theta)$. The tuple (p_L, p_H, θ) violates (MT) *if $e_H(1) < e_L(\theta)$.*

The next lemma provides an explicit equivalent condition and some illustrations of (MT). (MT) holds if high signals are more reliable than low ones, or if the overall reliability of signals is sufficiently high while the threshold θ is low. (MT) is violated if high signals are sufficiently unreliable, and if the threshold θ is sufficiently high.

Lemma 4

(i) (MT) *is equivalent to*

$$0 < S(p_L, p_H, \theta) = p_L + p_H + p_L p_H + \theta^2 p_L^2 - (1-\theta)^2 p_H^2 - 2\theta p_L - 2\theta p_L p_H.$$

(ii) (MT) *is satisfied if $p_H \geq p_L$.*

(iii) (MT) *is satisfied if $(1-\theta)(p_L + p_H) \geq 1$.*

(iv) For any $p_L \in (0,1)$, (MT) *is violated if p_H is sufficiently small and θ is sufficiently large.*

The next two propositions characterize the effects of the three types of information release, first for the case where (MT) holds and then for the case where it is violated. We indicate whether $F \succeq_{k,m} G$ or vice versa for sufficiently high m. The results on auctions then follow directly from Theorem 1.

Proposition 9 *Suppose (p_L, p_H, θ) satisfy* (MT).

(i) If F differs from G through a sufficiently small L-increase or T-increase in information, then for any k there exists m such that $F \succeq_{k,m} G$.

(ii) If F differs from G through a sufficiently small H-increase in information and if $p_H < \theta^{-1} - p_L$, then for any k there exists m such that $F \succeq_{k,m} G$.

(iii) If F differs from G through a sufficiently small H-increase in information and if $p_H > \theta^{-1} - p_L$, then for any k there exists m such that $G \succeq_{k,m} F$.

In the proposition, "a sufficiently small increase" means that the increase leaves condition (MT) intact and, in cases (ii) and (iii), also the additional restriction on p_H. Increasing the amount of information through changes in p_L or θ thus relaxes competition among sufficiently many bidders, i.e., assertions (i)-(iv) of Theorem 1 apply. In contrast, if θ, p_L and p_H are sufficiently high,[32] a further increase in p_H

[32]Notice that $p_H > \theta^{-1} - p_L$ can only hold if the right hand side is smaller than 1, i.e., if $(1 + p_L)\theta > 1$. To see that cases (ii) and (iii) of the proposition are both compatible with (MT),

induces a fiercer competition at the top and implies the reversals of assertions (i)-(iv). In the latter case a further increase in p_H leads to more bidders learning about their very high valuations. If the overall signal quality is already high, this effect dominates the welfare enhancing effects of information release such as a further differentiation of beliefs at the top.[33] Finally, we investigate the situation where (MT) is violated so that the highest bids come from bidders with signals slightly below θ.

Proposition 10 *Suppose* (p_L, p_H, θ) *violate* (MT). *If F differs from G through a sufficiently small H-increase, L-increase or T-increase in information, then for any k there exists m such that $F \succeq_{k,m} G$.*

Thus, if (MT) is violated and there are sufficiently many bidders, assertions (i)-(iv) of Theorem 1 hold for all three types of information release. Small amounts of information always soften competition at the top.

Our analysis describes which kind of information release appeals more to welfare-maximizing versus revenue-maximizing sellers. Another question is whether information release actually enhances welfare and the seller's revenue or not. In the information partitions model of Section 4.2, welfare and seller's revenue always increase in response to information release when there are sufficiently many bidders. In the model of this section, effects can be more intricate. With sufficiently many bidders, the question is equivalent to the question whether the upper end of the support $u = \max(e_L(\theta), e_H(1))$ increases in response to information release. When (MT) is satisfied, H- and L-increases in information always lead to an increase in $u = e_H(1)$ and thus to higher welfare and seller's revenue with sufficiently many bidders.[34]

consider $p = p_L = p_H > \frac{1}{2}$. Then (MT) holds by Lemma 4 and whether we are in case (ii) or (iii) depends on whether $\theta < (2p)^{-1} \in (0, 1)$ or not.

[33] In particular, the effect which leads to a reversal of Theorem 1 in this model is distinct from the one we observed in the case of information partitions. There, the increased competition at the top was due to a further differentiation of intermediate valuation estimates.

[34] For T-increases and for the case where (MT) is violated, the behavior of u is more complex and a detailed discussion is beyond the scope of this paper. The results of Theorem 1 remain valid when u decreases in response to information release but one might want to reinterpret (ii), e.g., in terms of incentives to prevent leakage of information.

5 Further Applications

This section sketches extensions of our analysis to other economic contexts, like matching markets, and the control of differences in low realizations which is important for risk management and reliability theory. Via k-dispersion, we can compare increments of expected order statistics $E[X_{k:n}]$ that are next to each other with regard to k or n. In this section, we show that k-dispersion serves as a tool for controlling differences of order statistics that lie further apart as well, and describe where this control can be applied.

Proposition 11 *If $F \succeq_k G$ for some $k < n$ then for all $i \leq k$ and all $l > i$*

$$E[X_{i:n} - X_{l:n}] \geq E[Y_{i:n} - Y_{l:n}].$$

Proposition 11 characterizes which degree of k-dispersion is needed in order to compare specific differences of order statistics. For example, the 1-dispersion order allows to contrast differences between first and third order statistics across distributions. A similar comparison of the second and third order statistics requires the stronger 2-dispersion order. The proposition generalizes the main result of Kochar, Li and Xu (2007)[35] which treats the case $k = 1$.

A direct consequence of Proposition 11 is that it allows to compare sums of spacings of order statistics: $F \succeq_k G$ implies that

$$\sum_{j=i}^{l-1} E[X_{j:n} - X_{j+1:n}] \geq \sum_{j=i}^{l-1} E[Y_{j:n} - Y_{j+1:n}]$$

for all $i \leq k$ and all $n > l > i$. Proposition 12 provides similar results for normalized spacings of order statistics.

Proposition 12 *If $F \succeq_k G$ for some $k < n$ then for all $i \leq k$ and all $l > i$*

$$\sum_{j=i}^{l} jE[X_{j:n} - X_{j+1:n}] \geq \sum_{j=i}^{l} jE[Y_{j:n} - Y_{j+1:n}].$$

[35] Kochar, Li and Xu apply their result to the study of one object q^{th} price auctions. This part of their analysis is problematic from the viewpoint of game-theoretic auction theory since it relies on the assumption that bidders truthfully bid their valuations independently of the auction format.

The case $k = 1$ generalizes a result of Barlow and Proschan (1966) which is a key ingredient of Hoppe, Moldovanu and Sela (2009)'s analysis of matching markets. In the latter paper, women and men can invest into costly presents in order to improve their matching outcomes (and thus, e.g., match with a partner that is ranked better than the partner they would obtain otherwise). The inequality of Proposition 12 allows to study the comparative statics of signaling costs and welfare in this marriage market. Barlow and Proschan (1966) rely on the convex transform order which is stronger than the excess wealth order when F and G have the same mean.[36] Proposition 12 shows that the results of Hoppe, Moldovanu, and Sela hold under weaker requirements on the distributions.

Regarding the spacings of the k lowest order statistics, one can define the family of $\overline{k}$-m-dispersion orders given by

$$F \succeq_{\overline{k}} G \quad \Leftrightarrow \quad \int_0^p u^k(1-u)^m dF^{-1}(u) \geq \int_0^p u^k(1-u)^m \, dG^{-1}(u) \quad \forall p \in (0,1).$$

For example, expected differences in quality for the worst, second to worst, third to worst, etc. product out of a production series can be compared through these orders. All arguments for this family of orders are parallel to those we obtained for the k-m- dispersion orders. Like the 1-0-dispersion order, the $\overline{1}$-0-dispersion order coincides with a familiar stochastic order, namely, with the location independent risk order of Jewitt (1989).

6 Conclusion

This paper has introduced new techniques for analyzing the impact of information release on revenues and welfare in independent private values auctions. From here, there are several avenues for further research. As sketched in the previous section, the results may inform various economic contexts such as matching markets or the study of economic inequality in which order statistics need to be handled. As the statistics and reliability theory literature inspired some of our techniques, our results may also prove useful in this domain. Finally, one can think of various challenging extensions to more general auction models. A generalization from independent

[36] Shaked and Shanthikumar (2007), formula (4.B.3), shows that the convex transform order implies the star order. Li (2005), Remark 2.7, shows that the star order implies the excess wealth order if F and G share the same mean.

private values to models with correlated valuations comes to mind. Further, one may want to think about models in which the auctioneer can send different signals to different bidders. This last point is particularly interesting since Bergemann and Pesendorfer (2007) have shown that – unless institutional requirements enforce symmetry – revenue-optimal information release consists of asymmetric information partitions.

A Proofs

Proof of Proposition 1

To see (i), notice that $F \succeq_{disp} G$ implies that the measure ν given by $d\nu(u) = d(F^{-1}(u) - G^{-1}(u))$ is non-negative, so that integrals of non-negative functions against ν are non-negative. Thus (2) holds for all p. (ii) is shown as follows: Lemma 7.1 of Chapter 4 of Barlow and Proschan (1981) states that for any signed measure ν on $\mathbb{R}^+$ and any non-decreasing, non-negative function h

$$\int_p^\infty d\nu(u) \geq 0 \quad \forall p > 0 \Rightarrow \int_0^\infty h(u) d\nu(u) \geq 0.$$

Applying this result with $d\nu(u) = (1-u)^{k+1} d(F^{-1}(u) - G^{-1}(u))$ shows that $F \succeq_{k+1} G$ implies

$$\int_0^1 h(u)(1-u)^{k+1} dF^{-1}(u) \geq \int_0^1 h(u)(1-u)^{k+1}\, dG^{-1}(u)$$

for any non-decreasing, non-negative h. Applying this inequality to all members of the family of non-decreasing functions $(h_q)_{q\in(0,1)}$ defined by $h_q(u) = (1-u)^{-1} 1_{\{u \geq q\}}$ yields

$$\int_q^1 (1-u)^k dF^{-1}(u) \geq \int_q^1 (1-u)^k\, dG^{-1}(u) \quad \forall q \in (0,1)$$

and thus $F \succeq_k G$. (iii) follows from the fact that $F \succeq_k G$ implies $F \succeq_1 G$ by (ii), and from the fact that $\succeq_1$ is the excess wealth order so that we can apply Formula 3.C.8 of Shaked and Shanthikumar (2007). □

Proof of Proposition 2

By Assertion (ii) of Proposition 1, it is sufficient to consider the case $k = i$. By (5), it is sufficient to show that

$$\int_0^1 u^{n-k}(1-u)^k dF^{-1}(u) \geq \int_0^1 u^{n-k}(1-u)^k dG^{-1}(u).$$

This inequality follows from the definition (2) of the k-dispersion order by applying – like in the proof of Proposition 1 – Lemma 7.1 of Chapter 4 of Barlow and Proschan (1981) to the signed measure ν given by $d\nu(u) = (1-u)^k d(F^{-1}(u) - G^{-1}(u))$ and to the non-decreasing function $h(u) = u^{n-k}$. $\square$

Proof of Proposition 3

Again, by Assertion (ii) of Proposition 1, it is sufficient to consider the case $k = i$. Rewriting Relation 1 from David (1970, p. 45) into our notation yields

$$E[X_{k:n}] - E[X_{k:n-1}] = \frac{k}{n}\,(E[X_{k:n}] - E[X_{k+1:n}]). \tag{9}$$

Thus we can apply Proposition 2 and conclude that $F \succeq_k G$ implies

$$\begin{aligned} E[X_{k:n}] - E[X_{k:n-1}] &= \frac{k}{n}(E[X_{k:n}] - E[X_{k+1:n}]) \\ &\geq \frac{k}{n}(E[Y_{k:n}] - E[Y_{k+1:n}]) = E[Y_{k:n}] - E[Y_{k:n-1}]. \end{aligned}$$

$\square$

Proof[37] of Proposition 4

By Proposition 5 (ii) we can focus on the case $i = k$. The proof of (i) is entirely parallel to the one of Proposition 2 except that we choose $d\nu(u) = u^m(1-u)^k d(F^{-1}(u) - G^{-1}(u))$ and $h(u) = u^{n-k-m}$. (ii) follows from (i) and (9). $\square$

Proof of Proposition 5

The proof of (i) is entirely parallel to the one of Proposition 1 (ii) except that we choose $d\nu(u) = u^m(1-u)^{k+1} d(F^{-1}(u) - G^{-1}(u))$. The same is true for the proof of (ii) where we choose $d\nu(u) = u^m(1-u)^k d(F^{-1}(u) - G^{-1}(u))$ and $h_q(u) = u1_{\{u\geq q\}}$. For (iii), notice that Proposition 8 provides a class of examples where $E[X_1] = E[Y_1]$, $F \succeq_{conv} G$ is satisfied together with either $F \succeq_{k,m} G$ or $G \succeq_{k,m} F$ for some m. $\square$

Proof of Proposition 6

Choose the measure ν as $d\nu(u) = (1-u)^k d(F^{-1}(u) - G^{-1}(u))$. We have to show that there exists m such that

$$L(r) = \int_r^1 u^m d\nu(u)$$

[37] The logical contingencies between Propositions 4 - 8 are as follows: Proposition 6 $\Rightarrow$ Proposition 8 $\Rightarrow$ Proposition 5 $\Rightarrow$ Proposition 4.

is non-negative for all $r \in (0,1)$. By assumption, the measure ν is nonnegative over $(p,1]$. This proves the claim for $r > p$. For $r \leq p$ consider the decomposition[38]

$$L(r) = \int_r^{p^+} u^m d\nu(u) + \int_{p^+}^{q_1} u^m d\nu(u) + \int_{q_1}^{q_2} u^m d\nu(u) + \int_{q_2}^{1} u^m d\nu(u).$$

The second and fourth integrals are non-negative by assumption. Since F^{-1} and G^{-1} are non-decreasing, we obtain the lower bound

$$L(r) \geq -\int_{0^+}^{p^+} u^m (1-u)^k dG^{-1}(u) + \int_{q_1}^{q_2} u^m d\nu(u).$$

Since both integrals are with respect to a nonnegative measure, we can further bound them by

$$L(r) \geq -p^m \int_{0^+}^{p^+} dG^{-1}(u) + q_1^m (1-q_2)^k \int_{q_1}^{q_2} d(F^{-1}(u) - G^{-1}(u)).$$

The right hand side equals $-p^m d_- + q_1^m (1-q_2)^k d_+$ which is non-negative for sufficiently large m since $(1-q_2)^k d_+ > 0$ and $q_1 > p$. To conclude the proof, it suffices to solve $-p^m d_- + q_1^m (1-q_2)^k d_+ \geq 0$ for m. $\square$

Proof of Corollary 1

The fact that f is strictly smaller than g at the top of the support implies that there exists a threshold p such that all pairs of quantiles greater than the p-quantile lie strictly further apart under F than under G. This implies $F_{>p} \succeq_{disp} G_{>p}$ and the claim follows from Proposition 6. $\square$

Proof of Proposition 7

Suppose that F takes values $x_1 > \ldots > x_{n_F}$ with positive probabilities $p_1, \ldots, p_{n_F}$ while G takes values $y_1 > \ldots > y_{n_G}$ with positive probabilities $q_1, \ldots, q_{n_G}$. Since the dispersive order is invariant under horizontal shifts, we can assume without loss of generality that $x_1 = y_1$, i.e., both distributions are shifted so that they have the same largest realization. Let $m \geq 1$ be the smallest integer such that $x_m \neq y_m$ or $p_m \neq q_m$.

[38] To make the choice of p^+ in the integral boundaries rigorous, one can read the integrals as integrals with respect to the Borel-measure induced by $F^{-1} - G^{-1}$, see Remark 2.2 in Broniatowski and Decurninge (2015).

Consider first the case where $x_m \neq y_m$. Since $x_1 = y_1$, this implies $m > 1$. Define

$$p = 1 - (p_1 + \ldots + p_{m-1} + \min(p_m, q_m)) \,.$$

Then the distributions $F_{>p}$ and $G_{>p}$ have atoms of identical sizes on the same values except that the location of the smallest atom differs. When $x_m < y_m$, we have $F_{>p} \succ_{disp} G_{>p}$, and when $x_m > y_m$, we have $G_{>p} \succ_{disp} F_{>p}$. Consider next the case where $x_m = y_m$ but $p_m > q_m$. This implies $m < n_G$. Define

$$p = 1 - (p_1 + \ldots + p_m) \,.$$

Then we have $G_{>p} \succ_{disp} F_{>p}$ since the distributions $F_{>p}$ and $G_{>p}$ are identical except that $F_{>p}$ has an atom of size p_m in $x_m = y_m$. Under $G_{>p}$, this probability mass of p_m is split into an atom of size q_m at y_m while the remaining mass of $p_m - q_m$ is located at y_{m+1} and (possibly) below. The case where $x_m = y_m$ but $p_m < q_m$ is analogous. □

Proof of Corollary 2

The implication $(i) \Rightarrow (ii) \Rightarrow (iii)$ follows from Proposition 6. It remains to show $(iii) \Rightarrow (i)$. Suppose that there exists k and m such that $F \succeq_{k,m} G$. This implies $F \succeq_{k,l} G$ for any $l > m$. For finite distributions which are not horizontal shifts of each other, $F \succeq_{k,m} G$ and $G \succeq_{k,m} F$ cannot hold simultaneously. Thus, $G \succeq_{k,l} F$ is violated by assumption for all $l \geq m$. $G_{>p} \succ_{disp} F_{>p}$ would imply $G \succeq_{k,L} F$ for all $L > M$ for some M. This is a contradiction. Therefore, Proposition 7 implies that there exists p such that $F_{>p} \succ_{disp} G_{>p}$. □

Proof of Theorem 1

Observe that we can write bidders' aggregate rents after information release as

$$\sum_{j=1}^{q} E[X_{j:n} - X_{q+1:n}] = \sum_{j=1}^{q} jE[X_{j:n} - X_{j+1:n}].$$

To the expression on the right hand side we can apply Proposition 4 and conclude

$$\sum_{j=1}^{q} E[X_{j:n} - X_{q+1:n}] \geq \sum_{j=1}^{q} E[Y_{j:n} - Y_{q+1:n}]$$

which is (i). Rearranging this inequality yields

$$\sum_{j=1}^{q} E[X_{j:n} - Y_{j:n}] \geq E[qX_{q+1:n} - qY_{q+1:n}]$$

which proves (ii). The welfare gains from adding an additional bidder when releasing information are given by $\sum_{j=1}^{q} E[X_{j:n} - X_{j:n-1}]$. This is greater than the corresponding quantity with Y in place of X by Proposition 4. This shows (iii). The claim about the expected selling price in (iv) follows from observing that Proposition 4 yields

$$E[X_{q+1:n} - X_{q+1:n-1}] \geq E[Y_{q+1:n} - Y_{q+1:n-1}]$$

provided that $F \succeq_{q+1,m} G$. The statement about the seller's payoff follows by multiplying this inequality with q. (v) follows by exchanging the roles of F and G.□

Proof of Proposition 8

Denote by α^* the largest element of $(\alpha_i)_i$ which is not included in $(\beta_i)_i$ and set $p = H(\alpha^*)$. We prove (i) by showing that $G_{>p} \succeq_{disp} F_{>p}$ and then invoking Proposition 6. Denote by $\beta_+^* > \beta_-^*$ the upper and lower neighbors of α^* in the sequence $(\beta_i)_i$. Observe that the distributions $F_{>p}$ and $G_{>p}$ are both discrete distributions concentrated on a finite number of values. In particular, since the two partitions are identical from $\beta_+^* \in (\alpha_i)_i$ on, the two distributions are identical except for the lowest value. For $F_{>p}$, the lowest possible realization l_F is the conditional mean of H over the set $[\alpha^*, \beta_+^*]$, while for $G_{>p}$ this lowest realization is the conditional mean l_G over $[\beta_-^*, \beta_+^*]$. Both occur with the same positive probability $(H(\beta_+^*) - H(\alpha^*))/(1-p)$. Clearly, we have $l_F > l_G$. Since this difference between the lowest realizations is the only difference of $F_{>p}$ and $G_{>p}$, it follows directly that $G_{>p} \succeq_{disp} F_{>p}$. Since all probabilities are strictly positive, we can also guarantee existence of q_1 and q_2 as required by Proposition 6.

The proof of (ii) proceeds similarly by showing that $F_{>p} \succeq_{disp} G_{>p}$. We set $p = H(\beta_B)$. Then $G_{>p}$ is a degenerate distribution which takes as its only value the conditional mean of H over $[\beta_B, b]$. $F_{>p}$ takes at least two values with positive probability, since the sequence (α_i) contains at least one element which is greater than β_B. We thus have $F_{>p} \succeq_{disp} G_{>p}$. □

Proof of Corollary 3

It suffices to note that the proof of Proposition 8 (ii) still goes through in this more general setting, switching the roles of F and G when $\beta_B > \alpha_A$. $\square$

Proof of Lemma 4

A direct calculation reveals that

$$e_H(1) - e_L(\theta) = \frac{S(p_L, p_H, \theta)}{2(1 + (p_L - p_H)(1-\theta))(1 + (p_H - p_L)\theta)}.$$

Since $|p_H - p_L| < 1$, the denominator is always positive and (i) follows. For (ii), note that S is concave in p_H so it suffices to verify $S(p_L, p_L, \theta) = 2p_L(1-\theta) > 0$ and

$$S(p_L, 1, \theta) = 2p_L + 2\theta - 4p_L\theta - \theta^2(1 - p_L^2) > 0.$$

The last claim follows from the facts that $S(p_L, 1, \theta)$ is concave in θ and that $S(p_L, 1, 1) = (1 - p_L)^2 > 0$ as well as $S(p_L, 1, 0) = 2p_L > 0$. For (iii), notice that S can be written as

$$S(p_L, p_H, \theta) = p_L(1-\theta)(1 + p_H) + p_H + \theta^2 p_L^2 - (1-\theta)^2 p_H^2 - \theta p_L - \theta p_L p_H.$$

Applying in the first summand the assumed inequality $p_L(1-\theta) \geq 1 - p_H(1-\theta)$, and rearranging, shows that S is bounded from below by the function

$$M(p_L, p_H, \theta) = -p_H^2(2-\theta)(1-\theta) + p_H(1 + \theta - p_L\theta) + 1 - p_L\theta(1 - p_L\theta).$$

Since M is concave in p_H, $M > 0$ follows from the positivity of $M(p_L, 0, \theta) = 1 - p_L\theta(1 - p_L\theta)$ and $M(p_L, 1, \theta) = \theta(4 - 2p_L - \theta + p_L^2\theta)$. For (iv), it suffices to notice that S is continuous and $S(p_L, 0, 1) = -p_L(1 - p_L) < 0$. $\square$

Proof of Proposition 9

Since G is a mixture of uniform distributions, it suffices to study how the value of the density at the highest valuation estimates reacts to changes in the parameters and then to apply Corollary 1. Since (MT) holds, the value of the density at the top is given by

$$T(p_L, p_H, \theta) = \frac{q_H}{e_H(1) - e_H(\theta)} = \frac{(1 + \theta(p_H - p_L))^2}{p_H}.$$

The relevant derivatives of T are given by

$$\frac{\partial T}{\partial p_L} = -\frac{2\theta(1 + (p_H - p_L)\theta)}{p_H}, \quad \frac{\partial T}{\partial \theta} = \frac{2(p_H - p_L)(1 + (p_H - p_L)\theta)}{p_H}$$

and

$$\frac{\partial T}{\partial p_H} = -\frac{(1-(p_H+p_L)\theta)(1+(p_H-p_L)\theta)}{p_H^2}.$$

Since $|p_H - p_L|\theta < 1$, $\frac{\partial T}{\partial p_L}$ is always negative, implying that $F \succeq_{k,m} G$ for sufficiently large m by Corollary 1. $\frac{\partial T}{\partial \theta}$ is negative when $p_L > p_H$ and positive when $p_H > p_L$, implying that $F \succeq_{k,m} G$ follows if θ is shifted into the direction of the smaller probability. The sign of $\frac{\partial T}{\partial p_H}$ depends on the sign of $1-(p_H+p_L)\theta$ as indicated in the proposition. □

Proof of Proposition 10

We only point out the differences to the proof of Proposition 9. Since (MT) is violated, the density at the top is now given by

$$T(p_L, p_H, \theta) = \frac{q_L}{e_L(\theta) - e_L(0)} = \frac{(1+(1-\theta)(p_L-p_H))^2}{p_L}.$$

The derivatives with respect to θ, p_H and p_L are given by

$$\frac{\partial T}{\partial p_H} = -\frac{2(1-\theta)(1+(p_L-p_H)(1-\theta))}{p_L}, \quad \frac{\partial T}{\partial \theta} = \frac{2(p_H-p_L)(1+(p_L-p_H)(1-\theta))}{p_L}$$

and

$$\frac{\partial T}{\partial p_L} = -\frac{(1-(p_H+p_L)(1-\theta))(1+(p_L-p_H)(1-\theta))}{p_L^2}.$$

The signs of the derivatives follow like in Proposition 9 except that we do not distinguish cases because a violation of (MT) implies $(p_H + p_L)(1-\theta) < 1$ by Lemma 4. □

Proof of Propositions 11 and 12

It is convenient to give a combined proof of the two propositions. By Assertion (ii) of Proposition 1, it is sufficient to consider the case $k = i$. From (5) we obtain that

$$E[X_{k:n} - X_{l:n}] = \int_0^1 \sum_{j=k}^{l-1} \binom{n}{j} u^{n-j}(1-u)^j dF^{-1}(u)$$

and

$$\sum_{j=k}^{l} jE[X_{j:n} - X_{j+1:n}] = \int_0^1 \sum_{j=k}^{l-1} j \binom{n}{j} u^{n-j}(1-u)^j dF^{-1}(u).$$

Obviously, the right hand sides coincide up to the factor j in the second sum. In the following, we denote this factor by $\varphi(j)$ and consider the choices $\varphi(j) = 1$ and

$\varphi(j) = j$. Now we claim the following:

Claim: For both, $\varphi(j) = 1$ and $\varphi(j) = j$, there exists a non-decreasing function h such that we can write

$$\sum_{j=k}^{l-1} \varphi(j) \binom{n}{j} u^{n-j}(1-u)^j = h(u)(1-u)^k.$$

Provided that this claim is true, the desired inequality follows from the definition (2) of the k-dispersion order by applying – like in the proof of Proposition 1 – Lemma 7.1 of Chapter 4 of Barlow and Proschan (1981) to the signed measure ν given by $d\nu(u) = (1-u)^k d(F^{-1}(u) - G^{-1}(u))$ and to the non-decreasing function h identified in the claim: We obtain

$$\int_0^1 h(u) d\nu(u) \geq 0.$$

and thus

$$\int_0^1 \sum_{j=k}^{l-1} \varphi(j) \binom{n}{j} u^{n-j}(1-u)^j dF^{-1}(u) \geq \int_0^1 \sum_{j=k}^{l-1} \varphi(j) \binom{n}{j} u^{n-j}(1-u)^j dG^{-1}(u).$$

Thus it remains to prove the claim. Since we can write

$$\sum_{j=k}^{l-1} \varphi(j) \binom{n}{j} u^{n-j}(1-u)^j = (1-u)^k \sum_{j=k}^{l-1} \varphi(j) \binom{n}{j} u^{n-j}(1-u)^{j-k},$$

this amounts to proving that

$$h(u) = \sum_{j=k}^{l-1} \varphi(j) \binom{n}{j} u^{n-j}(1-u)^{j-k}$$

is increasing in u for our two choices of $\varphi(j)$. The key idea is to rewrite h in terms of a Binomial$(n-k, 1-u)$ distribution. We can write

$$h(u) = \sum_{j=0}^{l-k-1} \varphi(k+j) \binom{n}{k+j} u^{n-k-j}(1-u)^j = \sum_{j=0}^{n-k} \Psi(j) \binom{n-k}{j} u^{n-k-j}(1-u)^j$$

where

$$\Psi(j) = \varphi(k+j) \frac{\binom{n}{k+j}}{\binom{n-k}{j}} 1_{\{j<l-k\}} = \varphi(k+j) \frac{n \cdot \ldots \cdot (n-k+1)}{(j+k) \cdot \ldots \cdot (j+1)} 1_{\{j<l-k\}}.$$

For our two choices of φ which yield, respectively $\varphi(k+j) = 1$ and $\varphi(k+j) = j+k$, $\Psi(j)$ is clearly a non-negative, non-increasing function. Now denote by $Z_{n-k,1-u}$ a random variable distributed according to the Binomial$(n-k, 1-u)$ distribution. From writing h as

$$h(u) = E[\Psi(Z_{n-k,1-u})]$$

we can see that h is non-decreasing in u since Ψ is non-increasing. $\square$

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