The Feedback Effect in Two-Sided Markets with Bilateral Investments

Deniz Dizdar, Benny Moldovanu and Nora Szech*

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Abstract

Agents in a finite two-sided market are matched assortatively, based on costly investments. Besides signaling privately known, complementary types, the investments also directly benefit the match partner. The bilateral external benefits induce a complex feedback cycle that amplifies the agents' signaling investments. Our main results quantify how the feedback effect depends on the numbers of competitors on both sides of the market. This yields detailed insights into the equilibria of two-sided matching contests with incomplete information, in particular for markets of small or intermediate size. It also allows us to shed some new light on the relationship between finite and continuum models of pre-match investment.

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*Dizdar: Department of Economics, University of Montréal, C.P. 6128 succursale Centre-Ville, Québec H3C 3J7, Canada. Phone: (+1) 514-343-2126. Email: deniz.dizdar@umontreal.ca. Moldovanu: Department of Economics, University of Bonn, Lennéstr. 37, 53113 Bonn, Germany. Phone: (+49) 228-73-9242. Email: mold@uni-bonn.de. Szech: Department of Economics and Management, Karlsruhe Institute of Technology, Kaiserstr. 12, 76131 Karlsruhe, Germany. Phone: (+49) 721-608-43809. Fax: (+49) 721-608 - 43807. Email: nora.szech@kit.edu.

1 Introduction

Signals play an important role whenever agents need to form matches. In virtually every real-life matching situation, be it in the labor, the marriage, or the education market, agents use signals in order to transmit information about their own quality. In addition to their signal value, such investments also yield direct utility to the matched partner. Flashy facilities built by universities or firms are signals of quality that also create direct benefits for future students or employees, while individuals' investments in education and training signal underlying ability and are also valued directly by universities or prospective employers.

We study two-sided matching contests with arbitrary numbers of participants and with an NTU (nontransferable utility) matching market. The investments used by agents to signal information about privately known, complementary productive types also benefit the match partner.

The bilateral external benefits induce a feedback cycle that may cause agents to invest much more than they would if signals were completely wasteful. Increased signaling on one side of the market (more precisely, larger differences between the investments of different types) intensifies the competition among agents on the other side, causing them to invest more, which in turn intensifies again the competition on the first side, and so on...

In this paper, we shed light on how the feedback cycle works in finite markets with incomplete information, under the standard assumption that agents on each side of the market are ex-ante symmetric. In particular, we quantify how the feedback cycle depends on the numbers of agents on both sides of market, and we examine how it affects agents' equilibrium behavior and interim expected utilities.

Our focus is on finite markets since many real life examples, be it the marriage market in a rural area, or the labor market for specialized workers, are best modeled by assuming a moderate market size. Also standard market experiments in the laboratory focus on markets of intermediate size.

Both functions of pre-match investments have been emphasized in the literature. Hoppe, Moldovanu and Sela (2009, henceforth HMS) analyze signaling behavior when the signals are *completely wasteful* as in Spence (1973). In this case, the investment feedback effect does not exist. On the other hand, in the important papers by Peters and Siow (2002), Peters (2007) and Bhaskar and Hopkins (2016), an agent's pre-match investment benefits his or her partner directly (so that feedback effects are present) but it has no signaling effect. Arguably, most pre-match investments observed in reality have both a signaling and a productive function (Hopkins 2012), which motivates our setting.

Our two-sided *matching contest* model combines the signaling model of HMS and the investment model of Peters (2007), and has the following main features:

1. There is a finite number of agents on either side of the market. Agents are called men and women, respectively. Men and women are characterized by privately known, linearly ordered types that are complementary in the production of output. Consequently, the matching that maximizes aggregate output is positively assortative in types. Signaling can be sustained, as in HMS, without assuming heterogeneity in investment costs. Types are drawn i.i.d. from two commonly known distributions. We study equilibria in which all agents on the same side of the market use the same, strictly increasing investment strategy, so that positively assortative matching based on observable investments implies positively assortative matching of types.

2. In contrast to the HMS model, signals are not completely wasteful: they generate benefits for partners that are increasing in the level of investment. As in the complete information setting of Peters (2007), we assume that these benefits enter agents' utilities additively.

In a market with finitely many participants, agents face uncertainty about the types of competitors. They are also uncertain about the actual types and investments of potential partners. These uncertainties differentiate our model from standard pre-match investment models with a continuum of heterogeneous agents, in which each agent knows exactly where he or she is ranked in the competition, and what he or she will get in return for any particular investment. Relative to a continuum economy, the finiteness of the market and the resulting uncertainties dampen the investment feedback effect. We quantify how the strength of the feedback effect depends on the numbers of men and women, and we identify the largest eigenvalue of a particular matrix as the key measure for the dampening of the feedback loop. More precisely, we derive sharp conditions for the existence of side-symmetric, strictly separating equilibria that only depend on this feedback coefficient (which is smaller than 1 in any finite market, while it is equal to 1 in a continuum economy) and on the parameters describing the marginal external benefits and the marginal costs of the investments. If the product of marginal external benefits is too high compared to the product of marginal costs and if competition is too intense, the feedback process can become self-perpetuating and push investments beyond any bounded multiple of the pure signaling investments.[1] For the case in which

[1]This is similar to the occurrence of "ruinous" gift exchanges, as documented in the anthropological

the external benefits to partners are linear functions of the investments, we also find the unique side-symmetric, separating equilibrium in closed form. For most practical cases existence is not an issue, but the increases in investment needed to signal small quality differences may be very large even in markets of moderate size. A related phenomenon currently seems to arise in the case of US colleges. The New York Times speaks of a 'paradox' which in our model, however, occurs in equilibrium:

> "Typically, fierce market competition leads to lower prices, but among elite schools, the opposite occurs, paradoxically. They often find that raising prices enables them to offer greater benefits to the most coveted potential students. (It also allows them to take part in the amenities race: nicer dorms, better food, a climbing wall: things that are regarded as essential to attracting those coveted students.)"[2]

Investments into students' amenities (and students' fees) thus steeply increase across competitors in order to signal (probably much smaller) differences in quality.

Computing the feedback coefficient explicitly (in closed form for balanced or slightly unbalanced markets, numerically for all other markets) allows us to obtain detailed qualitative and quantitative insights into how the strength of the feedback cycle depends on market size. For example, entry of additional agents on the short side of the market or simultaneous entry on both sides intensify the feedback cycle while, somewhat surprisingly, entry on the long side only generally weakens it. In particular, this implies that some of the main entry-related comparative statics results in HMS do not extend to the case of *partially wasteful* signals.

Our results also produce some interesting bounds on under-investment for environments in which investments are "truly" *productive* in the sense that all Pareto efficient and individually rational investments for a given pair of agents are strictly positive (i.e., exceed the privately optimal investments). These bounds provide quantitative information about the extent to which competition can rule out extreme under-investment in small markets with productive investments.

For matching contests in which investments are *partially wasteful*, we identify the exact asymptotic behavior of equilibrium utilities as the numbers of men and women go to infinity. In this case, equilibrium utilities converge to those in the unique equilibrium of a continuum model, for which the return to any possible investment is certain. In

literature following Boas (1897) and Mauss (1935): "A gives 10 blankets to B; after an interval of time B gives 20 blankets to A... and so it goes on with the number of blankets being given increasing at a geometric rate" (see Gregory, 1980).

[2]Davidson, Adam (2015): Is College Tuition Really Too High? In: New York Times, 08 Sept 2015. https://www.nytimes.com/2015/09/13/magazine/is-college-tuition-too-high.html

particular, this shows that, even though investments are only partially wasteful, the entire difference between aggregate match surplus and aggregate information rents gets dissipated through competition.

If the marginal benefit from investment is constant and equal to the marginal cost (called below the *transferable utility* or *TU* investment case), the continuum model does not admit a side-symmetric strictly separating equilibrium: the intense competition together with the certainty of returns drive investments to infinity. However, such an equilibrium exists in any finite market with the same characteristics and we are able to characterize the limit behavior of equilibrium utilities: in large, balanced markets, the difference between aggregate match surplus and aggregate information rents is always shared fifty-fifty between men and women, irrespectively of other economic aspects such as the shares governing the division of physical surplus in each matched pair and the distributions of types.

Related Literature

Considering one side of the market only, our agents are in a contest situation (see e.g. the survey of Konrad 2007): they compete by means of sunk investments for heterogeneous "prizes," which correspond to matches with the various potential partners. Recognizing this analogy, a sizeable literature has studied pre-match investment problems as matching contests, where agents on both sides of a two-sided market make observable investments and are then matched positive assortatively on the basis of these investments. In these papers, positively assortative matching based on investments is typically assumed, but it also corresponds to the stable outcome of a frictionless matching market (post-investment) with nontransferable utility. This is the case for the complete information models of Peters and Siow (2002), Peters (2007), and Bhaskar and Hopkins (2016), in which an agent who invests more generates higher benefits for partners, and also (in equilibrium) for the signaling model of HMS.

The challenges of analyzing non-cooperative equilibria of two-sided matching contests with a *finite* number of participants when external benefits generate feedback effects are succinctly described in Peters (2007).[3] With a few important exceptions (Peters 2007, 2011; Bhaskar and Hopkins 2016; Cole, Mailath and Postlewaite 2001b; Felli and Roberts 2016), the literature on pre-match investment problems has circumvented this difficulty by focusing on continuum models in which agents behave competitively (e.g., Cole, Mailath and Postlewaite 2001a; Peters and Siow 2002; Nöldeke

[3] Most of the contracting literature has focused on one pair in isolation, e.g. Che and Hausch (1999) who study the hold-up problem in a bilateral contracting situation with cooperative investments that benefit the partner.

and Samuelson 2015; Dizdar 2017).

The work of Peters (2007, 2011) demonstrates that non-cooperative equilibrium investments in very large (but finite) two-sided matching contests can be quite different from the investments predicted by a continuum model with competitive agents. More precisely, for models without signaling concerns and with productive investments, Peters shows that equilibrium investments in unbalanced matching contests generally do not converge to competitive (or hedonic) equilibrium investments as the numbers of men and women go to infinity. In particular, agents at the bottom of the distributions generally over-invest. Our results and techniques do not allow new insights about equilibria in very large markets with truly productive investments, but the arguments in Peters (2011) imply that qualitative results similar to those in his paper, with additional over-investment due to signaling, must hold in our model for this case (compare the discussion in Section 4). We focus instead on a much more detailed analysis of the feedback cycle due to external benefits in a model where investments also serve as signals.

Bhaskar and Hopkins (2016) study a model with an NTU matching market and *noisy* investments, building on the tournament model of Lazear and Rosen (1981) rather than on the literature on all-pay contests. Moreover, they assume complete information and that agents on either side of the market are ex-ante symmetric. They prove the existence of a unique equilibrium and show that agents over-invest unless the two sides of the market are completely symmetric. While their main focus is on the analysis of a continuum model, they also show (under certain conditions) that the corresponding, unique equilibrium is the limit of the non-cooperative equilibria for a finite model.[4]

Olszewski and Siegel (2016) characterize asymptotic bidding behavior in one-sided all-pay contests with many agents and many prizes. Their general results allow for complete or incomplete information and for ex-ante asymmetric agents, but because the prize structure is given exogenously these findings cannot be applied to characterize equilibrium behavior in environments with bilateral investments and with external benefits. Moreover, their results only hold for very large contests where particular approximation techniques can be applied.

[4]Cole, Mailath and Postlewaite (2001b) and Felli and Roberts (2016) are less directly related to the present study because they analyze models with a TU matching market (i.e., the division of joint surplus is fully flexible) and complete information. Cole, Mailath and Postlewaite (2001b) provide a (non-generic) condition on the ex ante heterogeneity of agents that ensures the existence of a Pareto efficient equilibrium, and they study potential coordination failures due to a form of market incompleteness. Felli and Roberts (2016) characterize the inefficiencies that arise, due to hold-up and coordination problems, when Cole, Mailath and Postlewaite's condition is violated, and when the matching is determined (post-investment) by a particular bidding game.

Outline

The paper is organized as follows. In Section 2, we introduce the model and define various pieces of notation. Section 3 presents the basic equilibrium characterization and our main results about the investment feedback effect, including the closed form solution for the case of linear external benefits and several illustrations. Section 4 contains the results for large markets with partially wasteful or TU investments. All proofs are in an Appendix.

2 Model

We consider a matching market with n men and k women, where $n \geq k \geq 2$. If $n = k$, we say that the market is *balanced*. Otherwise, it is *unbalanced*. Each man is characterized by a privately known type $m \in [\underline{m}, \overline{m}]$, and each woman is characterized by a privately known type $w \in [\underline{w}, \overline{w}]$, where $0 \leq \underline{m} < \overline{m} < \infty$ and $0 \leq \underline{w} < \overline{w} < \infty$. Types are drawn independently from two commonly known, continuous distributions F (for men) and G (for women) with densities f and g that are strictly positive and continuous on $[\underline{m}, \overline{m}]$ and $[\underline{w}, \overline{w}]$, respectively.

All agents simultaneously make costly investments. Men and women are then matched positive assortatively according to their investments: the man with the highest investment is matched to the woman with the highest investment, the man with the second highest investment is matched to the woman with the second highest investment, and so on. Ties are broken randomly.

The net utility of a man with type m and investment $\beta_M \in \mathbb{R}_+$ who is matched to a woman with type w and investment $\beta_W \in \mathbb{R}_+$ is

$$\gamma_M m w + \delta_M(\beta_W) - \beta_M,$$

and the net utility of the woman in this match is

$$\gamma_W m w + \delta_W(\beta_M) - \beta_W.$$

Here, $\gamma_M > 0$ and $\gamma_W = 1 - \gamma_M > 0$ are constants, and $\delta_M : \mathbb{R}_+ \to \mathbb{R}_+$ and $\delta_W : \mathbb{R}_+ \to \mathbb{R}_+$ are non-decreasing, concave, twice continuously differentiable and satisfy $\delta_M(0) = 0$ and $\delta_W(0) = 0$.[5] The net utility of an unmatched man (woman) with investment β_M

[5] Large parts of our basic equilibrium characterization, including the sufficient condition for existence in Theorem 2 (if the condition about limits is replaced by an analogous condition about limit superiors) apply for arbitrary non-decreasing functions δ_M and δ_W, but as all our other quantitative results pertain

(β_W) is given by $-\beta_M$ ($-\beta_W$).

Note that agents' utility functions are strictly supermodular in their own type and their partner's type, so that investments serve as costly signals in an environment satisfying the standard single-crossing property. HMS's model of assortative matching based on *completely wasteful* signaling corresponds to the case $\delta_M = \delta_W \equiv 0$. Following the complete information models of Peters (2007) and Bashkar and Hopkins (2016), the external benefits enter agents' utilities additively, and are modeled via the non-decreasing and type-independent functions δ_M and δ_W.[6]

2.1 Notation

$\mathbb{R}_+$ ($\mathbb{R}_{++}$) denotes the set of non-negative (strictly positive) real numbers. We represent vectors in a Euclidean space $\mathbb{R}^l$, $l \in \mathbb{N}$, with respect to a fixed orthonormal basis, and we label the coordinates $0, ..., l-1$. I_l is the identity on $\mathbb{R}^l$, and $\cdot$ is the standard inner product: for $u, v \in \mathbb{R}^l$, $u \cdot v = \sum_{i=0}^{l-1} u_i v_i$. For $u \in \mathbb{R}^l$, $||u||_\infty$ and $||u||_1$ denote the vector's l_∞-norm and l_1-norm, respectively.[7] $u > 0$ ($u \geq 0$) means that all entries of u are strictly positive (non-negative). Similarly, for a real matrix A, $A > 0$ ($A \geq 0$) means that all entries of A are strictly positive (non-negative). For a square matrix $A \in \mathbb{R}^{l \times l}$, $\rho(A)$ is the spectral radius of A, i.e.,

$$\rho(A) := \max\{|\lambda| : \lambda \in \mathbb{C} \text{ is an eigenvalue of } A\},$$

and $|||A|||_\infty := \max_{u \neq 0} \frac{||Au||_\infty}{||u||_\infty}$ denotes the matrix norm that is induced by $||\cdot||_\infty$.

We let

$$M_{1:n} \leq M_{2:n} \leq ... \leq M_{n:n}$$

and

$$W_{1:k} \leq W_{2:k} \leq ... \leq W_{k:k}$$

denote the order statistics of men's and women's types and write $F_{i:n}$ ($G_{i:k}$) and $f_{i:n}$ ($g_{i:k}$) for the c.d.f. and p.d.f. of $M_{i:n}$ ($W_{i:k}$). Thus,

$$F_{i:n}(m) = \sum_{l=i}^{n} \binom{n}{l} F(m)^l (1 - F(m))^{n-l}, \tag{1}$$

to linear or concave benefits, we focus on this case from the outset.

[6] We could easily replace $\gamma_M mw$ and $\gamma_W mw$ by arbitrary smooth and strictly supermodular functions. Such a generalization would require only minor changes to the present analysis. By contrast, weakening additive separability would substantially reduce the analytical tractability of the model.

[7] That is, $||u||_\infty = \max_{i \in \{0,...,l-1\}} |u_i|$ and $||u||_1 = \sum_{i=0}^{l-1} |u_i|$.

and

$$f_{i:n}(m) = n\binom{n-1}{i-1}F(m)^{i-1}(1-F(m))^{n-i}f(m), \tag{2}$$

and $G_{i:k}$ and $g_{i:k}$ are given by analogous formulas. For convenience, we also define $M_{0:n} \equiv 0$ and $W_{0:k} \equiv 0$, so that $F_{0:n} = G_{0:k}$ is the c.d.f. of a Dirac measure at 0.

Next, we define $\mathscr{G} : \mathbb{R} \to \mathbb{R}^k$ as

$$\mathscr{G}_j(w) := \begin{cases} G_{j:k-1}(w) & \text{if } j \in \{1,...,k-1\} \\ 0 & \text{if } j = 0. \end{cases}$$

For a woman with type $w \in [\underline{w}, \overline{w}]$, $G_{j:k-1}(w)$ is the probability that j or more out of the $k-1$ other women have a type below her own. The entry $\mathscr{G}_0(w) \equiv 0$ will be convenient for representing the fact that women do not have to compete for a match with the k-th highest type of man. Similarly, we define $\mathscr{F} : \mathbb{R} \to \mathbb{R}^n$ as

$$\mathscr{F}_j(m) := \begin{cases} F_{j:n-1}(m) & \text{if } j \in \{1,...,n-1\} \\ 0 & \text{if } j = 0. \end{cases}$$

For any function $h : \mathbb{R} \to \mathbb{R}$, we define $\Delta_M^h \in \mathbb{R}^n$ as the vector with entries

$$\Delta_{M,i}^h = E[h(M_{i+1:n}) - h(M_{i:n})] \quad \text{for } i \in \{0,...,n-1\}.$$

Similarly, $\Delta_W^h \in \mathbb{R}^k$ is the vector with entries

$$\Delta_{W,i}^h = E[h(W_{i+1:k}) - h(W_{i:k})] \quad \text{for } i \in \{0,...,k-1\}.$$ [8]

3 Equilibrium Characterization

In this section, we provide our main results about how the strength of the investment feedback effect depends on the numbers of men and women, including our closed-form solution for the case of linear benefits. We also illustrate several implications for equilibrium investments in small markets, highlighting differences to HMS's analysis for the case where investments are completely wasteful.

We focus on *side-symmetric, strictly separating Bayes-Nash equilibria*, i.e., equilibria where all men use the same, strictly increasing strategy and all women use the same,

[8] We suppress the dependence on n and k in the notation for various quantities, such as Δ_M^h and Δ_W^h. We will add superscripts in Section 4, where we study sequences of matching contests of different sizes.

strictly increasing strategy.[9] Any such equilibrium implements the surplus-maximizing, positive assortative matching of types. We denote equilibrium strategies by $b_M : [\underline{m}, \overline{m}] \to \mathbb{R}_+$ (for men) and $b_W : [\underline{w}, \overline{w}] \to \mathbb{R}_+$ (for women).

We first discuss some basics of the equilibrium characterization. Suppose that all women use the same, strictly increasing strategy b_W. Then, the investment game among men is equivalent to an all-pay auction with incomplete information, n ex-ante symmetric bidders and k heterogeneous prizes. If $n = k$, every man is guaranteed to get at least a match with the worst-ranked partner. For a man with type m, the expected utility from this match (the "k-th prize") is

$$\gamma_M m E[W_{1:k}] + E[\delta_M(b_W(W_{1:k}))].$$

The matches with the better-ranked partners then correspond to the $k-1$ prizes that men actually compete for. In particular, for a man with type m the increase in expected utility associated with getting a match with the partner of type $W_{i+1:k}$ rather than with $W_{i:k}$, $i \in \{1, ..., k-1\}$, is

$$\gamma_M m E[W_{i+1:k} - W_{i:k}] + E[\delta_M(b_W(W_{i+1:k})) - \delta_M(b_W(W_{i:k}))],$$

which is strictly increasing in m (strict single crossing). Note how the prizes here are endogenous and depend on the strategy employed by women. This is the defining characteristic of two-sided contests with investments that generate external benefits.

If $n > k$, men also have to compete for the "k-th prize". In either case, existing results for all-pay auctions with incomplete information and with ex-ante symmetric bidders imply that the contest among men has a unique symmetric equilibrium. Moreover, the strictly increasing, differentiable equilibrium strategy can be derived by the standard first-order approach.[10] An analogous argument applies for women, who are guaranteed to get at least a match with the k-th ranked man, $M_{n-k+1:n}$.

Thus, a side-symmetric, strictly separating equilibrium of the matching contest corresponds to a pair of functions: b_M is the symmetric equilibrium strategy of the all-pay auction among men for which the prizes are determined by the order statistics induced by G and by the women's strategy b_W, and b_W is the symmetric equilibrium strategy of the all-pay auction among women for which the prizes are induced by the order statistics of F and by the men's strategy b_M. In equilibrium, the types are matched positive

[9]If $n > k$, strict supermodularity and the fact that men want to avoid staying unmatched ensures that all side-symmetric equilibria are strictly separating (given that positive assortative matching is assumed as part of the game). If $n = k$, there is also a side-symmetric equilibrium in which nobody invests.

[10]For details on sufficiency, see Moldovanu and Sela (2001), Appendices A and C.

assortatively: for $i \in \{1, ..., k\}$, the man with type $M_{n-k+i:n}$ is matched to the woman with type $W_{i:k}$.

A man with type $\underline{m}$ gets the worst possible match with probability 1 in equilibrium,[11] which implies $b_M(\underline{m}) = 0$.[12] Similarly, $b_W(\underline{w}) = 0$.[13] For a man with type $m > \underline{m}$ who assumes that all other agents use strictly increasing, differentiable strategies b_M and b_W, the problem of maximizing his expected utility is to choose an $s \in [\underline{m}, \overline{m}]$ that maximizes

$$\sum_{j=0}^{k-1} \mathscr{F}_{n-k+j}(s) E[\gamma_M m(W_{j+1:k} - W_{j:k}) + \delta_M(b_W(W_{j+1:k})) - \delta_M(b_W(W_{j:k}))] - b_M(s)$$
$$= \hat{\mathscr{F}}(s) \cdot (\gamma_M m \Delta_W^{I_1} + \Delta_W^{\delta_M \circ b_W}) - b_M(s), \tag{3}$$

where we have used the following notation.

Definition 1. *For $u \in \mathbb{R}^n$, let $\hat{u} \in \mathbb{R}^k$ denote the vector with entries*

$$\hat{u}_i = u_{n-k+i} \text{ for } i \in \{0, ..., k-1\}.$$

In equilibrium, the first order condition must be satisfied at $s = m$, i.e.,

$$b_M'(m) = \hat{\mathscr{F}}'(m) \cdot (\gamma_M m \Delta_W^{I_1} + \Delta_W^{\delta_M \circ b_W}).$$

Integrating the above, we obtain

$$b_M(m) = \gamma_M a_M(m) + \hat{\mathscr{F}}(m) \cdot \Delta_W^{\delta_M \circ b_W}, \tag{4}$$

where

$$\gamma_M a_M(m) = \gamma_M \left(m\hat{\mathscr{F}}(m) - \int_{\underline{m}}^{m} \hat{\mathscr{F}}(s) ds \right) \cdot \Delta_W^{I_1}$$

is men's equilibrium strategy in the HMS special case where women's investments create no benefits for them ($\delta_M \equiv 0$). In the sequel, we often refer to $\gamma_M a_M$ as men's *pure signaling investments*. An analogous derivation yields

$$b_W(w) = \gamma_W a_W(w) + \mathscr{G}(w) \cdot \hat{\Delta}_M^{\delta_W \circ b_M}, \tag{5}$$

where

$$a_W(w) = \left(w\mathscr{G}(w) - \int_{\underline{w}}^{w} \mathscr{G}(s) ds \right) \cdot \hat{\Delta}_M^{I_1}.$$

[11] If $n = k$, he is matched to the woman with type $W_{1:k}$. If $n > k$, he stays unmatched.

[12] If $b_M(\underline{m})$ were strictly positive, type $\underline{m}$ could decrease his investment without changing his expected match.

[13] If $\underline{w} > 0$, we also formally set $b_W(0) = 0$ (so that $b_W(W_{0:k}) = 0$).

In particular, using (4) we have for all $i \in \{0,...,n-1\}$:

$$\Delta_{M,i}^{\delta_W \circ b_M} = E[\delta_W(\gamma_M a_M(M_{i+1:n}) + \hat{\mathscr{F}}(M_{i+1:n}) \cdot \Delta_W^{\delta_M \circ b_W}) \\ - \delta_W(\gamma_M a_M(M_{i:n}) + \hat{\mathscr{F}}(M_{i:n}) \cdot \Delta_W^{\delta_M \circ b_W})]. \tag{6}$$

Similarly, (5) implies for all $i \in \{0,...,k-1\}$:

$$\Delta_{W,i}^{\delta_M \circ b_W} = E[\delta_M(\gamma_W a_W(W_{i+1:k}) + \mathscr{G}(W_{i+1:k}) \cdot \hat{\Delta}_M^{\delta_W \circ b_M}) \\ - \delta_M(\gamma_W a_W(W_{i:k}) + \mathscr{G}(W_{i:k}) \cdot \hat{\Delta}_M^{\delta_W \circ b_M})]. \tag{7}$$

Defining $T : \mathbb{R}_+^k \to \mathbb{R}_+^n$ and $S : \mathbb{R}_+^k \to \mathbb{R}_+^k$ via

$$T_i(y) := E[\delta_W(\gamma_M a_M(M_{i+1:n}) + \hat{\mathscr{F}}(M_{i+1:n}) \cdot y) - \delta_W(\gamma_M a_M(M_{i:n}) + \hat{\mathscr{F}}(M_{i:n}) \cdot y)],$$
$$S_i(x) := E[\delta_M(\gamma_W a_W(W_{i+1:k}) + \mathscr{G}(W_{i+1:k}) \cdot x) - \delta_M(\gamma_W a_W(W_{i:k}) + \mathscr{G}(W_{i:k}) \cdot x)],$$

we obtain the following fixed point characterization of side-symmetric, strictly separating equilibria of the matching contest.[14]

Lemma 1. *The mapping $\iota : b_W \mapsto \Delta_W^{\delta_M \circ b_W}$ is a bijection between the set of side-symmetric, strictly separating equilibria and the set of fixed points of $S \circ \hat{T}$.*[15]

The logic behind this fixed point argument is as follows. The equilibrium strategies determine the vectors of increments in the matching benefits $\Delta_W^{\delta_M \circ b_W}$ and $\hat{\Delta}_M^{\delta_W \circ b_M}$. Conversely, these increment vectors are sufficient to recover the strategies through (4) and (5). This is due to the fact that the increment vector of each market side determines the "prize structure" of the all-pay auction played by the other market side. In particular, the mappings S and $\hat{T}$ capture how any increment vector for one market side pins down a unique increment vector for the other side. In equilibrium, the pair of male and female increment vectors must be consistent with each other. In other words, $\Delta_W^{\delta_M \circ b_W}$ is a fixed point of $S \circ \hat{T}$ and $\hat{\Delta}_M^{\delta_W \circ b_M}$ is a fixed point of $\hat{T} \circ S$.

We now turn to the observation that will allow quantitative insights into equilibrium investment behavior. Men face the kind of uncertainty regarding the types of their competitors (and hence about which of the k prizes they will win with any particular investment) that is standard in auctions or in contests with one-dimensional heterogeneity and ex-ante symmetric agents. Interestingly, key aspects of how the heterogeneity of

[14] T maps $\mathbb{R}_+^k$ into $\mathbb{R}_+^n$ because δ_W, a_M and all coordinate functions of $\hat{\mathscr{F}}$ are non-decreasing. An analogous observation applies to S.

[15] Note that if b_W is an equilibrium strategy, men's equilibrium strategy is uniquely determined by (4).

the prizes affects the *investment increments* given by

$$\begin{aligned}\Delta_{M,i}^{b_M} &= E[b_M(M_{i+1:n}) - b_M(M_{i:n})] \\ &= \gamma_M \Delta_{M,i}^{a_M} + \sum_{j=0}^{k-1} E[F_{n-k+j:n-1}(M_{i+1:n}) - F_{n-k+j:n-1}(M_{i:n})] \Delta_{W,j}^{\delta_M \circ b_W}, \qquad (8)\end{aligned}$$

turn out to be *independent* of the distribution F (an analogous observation applies for women, of course). Indeed, $F(M_{i:n})$ is distributed like the i-th order statistic of n independent draws from the uniform distribution $U(0,1)$ (Theorem 1.2.5 in Reiss 1989), and $F_{j:n-1}$ is a polynomial in F. In particular, $E[F_{j:n-1}(M_{i:n})]$ does not depend on the distribution F (the term is a weighted sum of moments of the i-th order statistic of n independent draws from $U(0,1)$). This motivates the following definitions.

Definition 2. *For any $n \geq 2$, let $\Theta_n \in \mathbb{R}^{n \times n}$ be the matrix with entries*

$$\theta_{n,ij} = E[F_{j:n-1}(M_{i+1:n}) - F_{j:n-1}(M_{i:n})] \;\; for \;\; i,j \in \{0,...,n-1\}.$$

For any $n \geq k \geq 2$, let $\Theta_{n,k} \in \mathbb{R}^{n \times k}$ be the submatrix consisting of the last k columns of Θ_n, and let $\hat{\Theta}_{n,k}$ be the lower right (trailing) principal $k \times k$ submatrix of Θ_n.[16]

Armed with Definition 2, we can write (8) in the following compact form:

$$\Delta_M^{b_M} = \gamma_M \Delta_M^{a_M} + \Theta_{n,k} \Delta_W^{\delta_M \circ b_W}. \qquad (9)$$

Given the uncertainty inherent in a contest with n contestants and k prizes, the distribution-independent matrix $\Theta_{n,k}$ captures how the type-independent increments in expected benefits obtained by men from partners with higher investments translate into investment increments.[17]

An analogous remark applies of course for the vector of women's investment increments and to the matrix Θ_k, because

$$\Delta_{W,i}^{b_W} = \gamma_W \Delta_{W,i}^{a_W} + \sum_{j=0}^{k-1} E[G_{j:k-1}(W_{i+1:k}) - G_{j:k-1}(W_{i:k})] \hat{\Delta}_{M,j}^{\delta_W \circ b_M}$$

implies

$$\Delta_W^{b_W} = \gamma_W \Delta_W^{a_W} + \Theta_k \hat{\Delta}_M^{\delta_W \circ b_M}. \qquad (10)$$

[16] That is, the entries of $\hat{\Theta}_{n,k}$ are $\theta_{n,(n-k+i)(n-k+j)}$, for $i,j \in \{0,...,k-1\}$.

[17] Lemma 4 in the Appendix shows that the matrix $\Theta_{n,k}$ also occurs in the explicit representation of $\Delta_M^{a_M}$, i.e., it is also important for computing bid increments for the standard all pay-auction model with heterogenous prizes, where an agent's valuation is multiplicative in his/her type and the prize he/she gets.

The following lemma provides the explicit form of Θ_n, and establishes a somewhat surprising fact: $\frac{n+1}{n-1}\Theta_n$ is a stochastic matrix.

Lemma 2. *i) For all $i \in \{0,...,n-1\}$,*

$$\theta_{n,ij} = \begin{cases} 0 & \text{if } j = 0 \\ \frac{n-1}{2n-1}\frac{\binom{n}{i}\binom{n-2}{j-1}}{\binom{2n-2}{i+j-1}} & \text{if } j \in \{1,...,n-1\}. \end{cases}$$

ii) $\frac{n+1}{n-1}\Theta_n$ is a (row-)stochastic matrix.

If the benefit functions are linear, i.e., if $\delta_M(\beta_W) = d_M\beta_W$ and $\delta_W(\beta_M) = d_W\beta_M$ for constants $d_M, d_W \geq 0$, we have a precise characterization of the feedback cycle: if women's investment increments change by some $y \in \mathbb{R}^k$, men respond in a way that alters their investment increments by $\Theta_{n,k}d_M y$, which in turn (i.e., after one round through the feedback cycle) entails a further change of women's investment increments by $\Theta_k d_W \hat{\Theta}_{n,k} d_M y$. The total effect on women's investment increments induced by the entire feedback process (including the initial change) is therefore $\sum_{l=0}^{\infty}(d_M d_W \Theta_k \hat{\Theta}_{n,k})^l y$.

Definition 3. *Let $r(n,k) > 0$ be the* Perron root *of the non-zero and non-negative matrix $\Theta_k\hat{\Theta}_{n,k}$, i.e., the real eigenvalue attaining the spectral radius, $r(n,k) = \rho(\Theta_k\hat{\Theta}_{n,k})$.*[18]

Since Lemma 2 implies that $\Theta_k\hat{\Theta}_{n,k}$ is sub-stochastic, the Perron root $r(n,k)$ is smaller than one. It measures by how much the maximal strength of the feedback loop is dampened relative to a continuum economy where the analogous coefficient is $r = 1$ (see Section 4).

Theorem 1 (Equilibrium characterization for linear benefits)**.** *Assume that $\delta_M(\beta_W) = d_M\beta_W$ and $\delta_W(\beta_M) = d_W\beta_M$ for constants $d_M, d_W \geq 0$. A side-symmetric, strictly separating equilibrium exists if and only if $\rho(d_M d_W \Theta_k\hat{\Theta}_{n,k}) = d_M d_W r(n,k) < 1$. If it exists, the side-symmetric, strictly separating equilibrium is unique. The equilibrium strategies satisfy*

$$b_M(m) = \gamma_M a_M(m) + d_M \hat{\mathscr{F}}(m) \cdot \Delta_W^{b_W}$$

$$b_W(w) = \gamma_W a_W(w) + d_W \mathscr{G}(w) \cdot \hat{\Delta}_M^{b_M},$$

where $\Delta_W^{b_W}$ and $\hat{\Delta}_M^{b_M}$ are explicitly given by

$$\hat{\Delta}_M^{b_M} = (I_k - d_M d_W \hat{\Theta}_{n,k}\Theta_k)^{-1}(\gamma_M\hat{\Delta}_M^{a_M} + d_M\gamma_W\hat{\Theta}_{n,k}\Delta_W^{a_W}) \quad (11)$$

$$\Delta_W^{b_W} = (I_k - d_M d_W \Theta_k\hat{\Theta}_{n,k})^{-1}(\gamma_W\Delta_W^{a_W} + d_W\gamma_M\Theta_k\hat{\Delta}_M^{a_M}). \quad (12)$$

[18]The existence of $r(n,k)$ follows from Theorems 8.1.22 and 8.3.1 in Horn and Johnson (2013).

The condition $d_M d_W r(n,k) < 1$ says that the product of the two marginal external benefits from investment and of the feedback coefficient $r(n,k)$ is less than the product of marginal costs, which is normalized here to unity. In this case, the series $\sum_{l=0}^{\infty}(d_M d_W \Theta_k \hat{\Theta}_{n,k})^l$ converges, and is then equal to $(I_k - d_M d_W \Theta_k \hat{\Theta}_{n,k})^{-1}$, so that the feedback process triggered by men's and women's pure signaling investments converges. This is precisely the interpretation of the expression (12), because $\gamma_W \Delta_W^{a_W} + d_W \gamma_M \Theta_k \hat{\Delta}_M^{a_M}$ would be the vector of women's investment increments if women competed for men who make pure signaling investments. Observe also that the resulting equilibrium investments stay within a bounded multiple of the pure signaling investments.

If $d_M d_W r(n,k) \geq 1$, the feedback effect is so strong that the investments grow out of bounds and the studied equilibrium type ceases to exist.[19]

To better understand the significance of $r(n,k)$, consider the case $n > k$ and note that $\Theta_k \hat{\Theta}_{n,k}$ is positive.[20] By Perron's Theorem (Theorem 8.2.8 in Horn and Johnson 2013), the Perron root is the only eigenvalue of maximal modulus, and there are unique right and left corresponding eigenvectors $y(n,k) > 0$ and $w(n,k) > 0$, normalized such that $\|y(n,k)\|_1 = 1$ and $y(n,k) \cdot w(n,k) = 1$.[21] Moreover, the matrices $(r(n,k)^{-1}\Theta_k \hat{\Theta}_{n,k})^l$ converge to $y(n,k)w(n,k)^T$ as $l \to \infty$, and the error decays geometrically in l. Therefore, if $d_M d_W r(n,k)$ approaches 1, the dominating effect of the feedback process is amplifying the vector $\left(w(n,k) \cdot (\gamma_W \Delta_W^{a_W} + d_W \gamma_M \Theta_k \hat{\Delta}_M^{a_M})\right) y(n,k)$ by a factor of $1/(1 - d_M d_W r(n,k))$.[22]

Using some results from matrix analysis and Brouwer's Fixed Point Theorem, we also get a sharp condition for the existence of a side-symmetric, strictly separating equilibrium in the case of general, concave benefit functions.

Theorem 2. *For the general model of Section 2, a side-symmetric, strictly separating equilibrium exists if and only if* $(\lim_{b\to\infty} \delta_M'(b))(\lim_{b\to\infty} \delta_W'(b)) r(n,k) < 1$.

We conclude this section with a closer look at the values of the feedback coefficient $r(n,k)$. By Lemma 2, the matrices $\left(\frac{n+1}{n-1}\Theta_n\right)^2$, $\frac{n+1}{n-1}\frac{n+2}{n}\Theta_n \hat{\Theta}_{n+1,n}$ and $\left(\frac{n+2}{n}\Theta_{n+1}\right)^2$ are

[19] The proof for this case is based on Farkas' Lemma and uses the left eigenvector associated with the eigenvalue $r(n,k)$.

[20] If $n = k$, the first column of $\Theta_k \hat{\Theta}_{n,k} = \Theta_n^2$ is zero while the lower right $(n-1) \times (n-1)$ principal submatrix is positive, and the following explanations must then essentially be applied to this submatrix.

[21] That is, $\Theta_k \hat{\Theta}_{n,k} y(n,k) = r(n,k) y(n,k)$ and $w(n,k)^T \Theta_k \hat{\Theta}_{n,k} = r(n,k) w(n,k)^T$.

[22] Observe that $w(n,k) \cdot (\gamma_W \Delta_W^{a_W} + d_W \gamma_M \Theta_k \hat{\Delta}_M^{a_M}) > 0$ as both vectors lie in the positive orthant.

stochastic. This implies that for any $n \geq 2$:

$$r(n,n) = \left(\frac{n-1}{n+1}\right)^2 < r(n+1,n) = \frac{(n-1)n}{(n+1)(n+2)} < r(n+1,n+1) = \left(\frac{n}{n+2}\right)^2. \tag{13}$$

Thus, the strength of the feedback effect increases if one more agent enters a balanced market, or if an additional agent enters on the short side of a slightly unbalanced market.

If $k \leq n-2$, the row (and column) sums of $\Theta_k \hat{\Theta}_{n,k}$ are different, so that a closed-form expression for $r(n,k)$ does not exist. However, $r(n,k)$ can easily be computed numerically if k is not too large.[23] Table 1 below, which gives the values of $r(n,k)$ for $k \leq n \leq 14$ and also the limits $\lim_{n\to\infty} r(n,k)$ for $k \leq 14$,[24] already displays the general pattern.[25]

Table 1: Values of $r(n,k)$

n\k	2	3	4	5	6	7	8	9	10	11	12	13	14
2	0.1111												
3	0.1667	0.25											
4	0.1714	0.3	0.36										
5	0.1720	0.3035	0.4	0.4444									
6	0.1717	0.3028	0.4023	0.4762	0.5102								
7	0.1713	0.3014	0.4007	0.4777	0.5357	0.5626							
8	0.1709	0.3000	0.3986	0.4757	0.5367	0.5833	0.6049						
9	0.1706	0.2989	0.3966	0.4732	0.5346	0.5840	0.6222	0.64					
10	0.1703	0.2978	0.3949	0.4709	0.5320	0.5818	0.6226	0.6545	0.6694				
11	0.1700	0.2970	0.3934	0.4689	0.5295	0.5792	0.6206	0.6548	0.6818	0.6944			
12	0.1698	0.2962	0.3921	0.4671	0.5273	0.5768	0.6181	0.6528	0.6820	0.7051	0.7160		
13	0.1696	0.2956	0.3910	0.4655	0.5253	0.5745	0.6156	0.6504	0.6801	0.7052	0.7253	0.7347	
14	0.1694	0.2950	0.3900	0.4641	0.5236	0.5725	0.6134	0.6481	0.6779	0.7035	0.7253	0.7429	0.7511
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
∞	0.1667	0.2871	0.3766	0.4455	0.5003	0.5449	0.5819	0.6132	0.6400	0.6632	0.6835	0.7014	0.7173

First, $r(n,k)$ increases with k, i.e., adding an additional agent on the short side of an unbalanced market increases the strength of the feedback effect. Somewhat surprisingly, the effect of increasing n for a fixed value of k is non-monotonic. Adding one more man to a balanced market increases r (see (13); in Table 1, this can be seen from the entries on the main diagonal and the first subdiagonal), but entry on the long side of an already unbalanced market generally *decreases* the strength of the feedback effect. More precisely, the numerical results show that if $k \geq 14$ then $r(n,k)$ is decreasing in n from $n = k+1$ onwards, while the decreasing part starts a bit later for smaller values of k

[23]Fast algorithms (exploiting the special features of positive matrices) that compute the Perron root exist even for fairly high-dimensional matrices.

[24]For any fixed k, $\lim_{n\to\infty} \Theta_k \hat{\Theta}_{n,k}$ can easily be found using Lemma 2.

[25]We have to rely on numerical results for the following discussion. Existing upper and lower bounds for the Perron root of a positive matrix (see Chapter 8 in Horn and Johnson 2013) are not sharp enough to prove the corresponding monotonicity properties analytically. We have checked all values $k \leq n \leq 100$.

(at $n = k+3$ for $k = 2$ and at $n = k+2$ for $3 \leq k \leq 13$). The decrease of $r(n,k)$ becomes quite substantial as n grows large. In particular, the feedback coefficient can be larger in a market with strictly fewer agents on both sides (e.g., we have $r(13,12) > r(n,14)$ for large values of n).

3.1 Applications

In this Section we offer several applications of our insights above.

3.1.1 The Effect of Entry on Total Investments

HMS (see Proposition 4) showed that if investments are completely wasteful ($d_M = d_W = 0$), then entry of additional men (i.e., on the long side of the market) has an unambiguous effect on men's total investments:

$$TI_M(n,k) = \sum_{i=1}^{n} E[b_M(M_{i:n})] = \sum_{i=1}^{n} \gamma_M E[a_M(M_{i:n})].$$

is always increasing in n for any fixed k. This represents the familiar intuition that competition among men becomes stiffer when there are more competitors, but it is no longer true if, due to the feedback loop, the "prizes" are endogenous.

To illustrate this phenomenon, we focus here on the case of partially, but not completely wasteful investments. We say that investments are *partially wasteful* if $\delta'_M(0)\delta'_W(0) < 1$. The reason for this terminology is as follows. If $\delta'_M(0) > 0$ and $\delta'_W(0) > 0$, investments generate benefits for partners but as long as $\delta'_M(0)\delta'_W(0) < 1$, the only pair of Pareto efficient and individually rational investments for a given pair of agents[26] is $(\beta_M, \beta_W) = (0,0)$, and all $(\beta_M, \beta_W) \in R^2_{++}$ are inefficient.

As we have seen at the end of the previous section, the feedback coefficient *decreases* if agents enter on the long side of an already unbalanced market (with the minor exceptions for $k \leq 13$, mentioned above). As a consequence, men's total investments may actually be lower if there are more men, provided that the feedback effect is sufficiently strong. This is the case if the product of marginal external benefits is close to 1 and if k is not too small. For example, if $F = G = U[0,1]$ and $\gamma_M = \frac{1}{2}$, we find $TI_M(51,50) = 4.17 < TI_M(100,50) = 8.21 < TI_M(200,50) = 10.34$ for $d_M = d_W = 0$, but $TI_M(51,50) = 90.1 > TI_M(100,50) = 81.5 > TI_M(200,50) = 79.6$ for constant marginal external benefits $d_M = d_W = 0.98$.

[26] Two agents who are already matched before they invest and can write complete contracts would make such investments.

Proposition 4 in HMS also shows that if $d_M = d_W = 0$ and the distribution F has an increasing failure rate, then women's total investments

$$TI_W(n,k) = \sum_{i=1}^{k} E[b_W(W_{i:k})]$$

are always decreasing in n. Again, this is no longer true for partially wasteful investments: as the feedback effect becomes stronger if an additional man enters a balanced market, women's total investments may increase in this case. Table 2 illustrates this effect for $F = G = U[0,1]$, $\gamma_M = \frac{1}{2}$ and constant marginal benefits $d_M = d_W = 0.9$. The table, which depicts $TI_W(n,k)$ for $k \leq n \leq 14$, also illustrates how well the pattern of women's total investments follows the one of the feedback coefficient $r(n,k)$ in Table 1.

Table 2: $TI_W(n,k)$ for $F = G = U[0,1]$, $d_M = d_W = 0.9$ and $\gamma_M = \frac{1}{2}$

n\k	2	3	4	5	6	7	8	9	10	11	12	13	14
2	0.0913												
3	0.1015	0.272											
4	0.1018	0.290	0.530										
5	0.1007	0.290	0.554	0.855									
6	0.0995	0.287	0.552	0.883	1.237								
7	0.0984	0.284	0.547	0.878	1.269	1.669							
8	0.0974	0.281	0.541	0.869	1.260	1.704	2.146						
9	0.0965	0.279	0.536	0.860	1.247	1.690	2.182	2.661					
10	0.0957	0.276	0.531	0.852	1.234	1.673	2.164	2.698	3.21				
11	0.0951	0.274	0.527	0.845	1.223	1.656	2.142	2.675	3.25	3.79			
12	0.0945	0.273	0.523	0.838	1.213	1.641	2.121	2.649	3.22	3.83	4.39		
13	0.0940	0.271	0.520	0.832	1.203	1.628	2.102	2.624	3.19	3.80	4.44	5.02	
14	0.0936	0.270	0.517	0.827	1.195	1.616	2.085	2.601	3.16	3.76	4.40	5.07	5.68

3.1.2 The Effect of Changes in the Intra-Household Bargaining Power

We show here that if investments generate external benefits, the expected utility of low type agents may be *decreasing* in their share γ_i, $i \in \{M, W\}$, of the type-dependent output mw. This phenomenon cannot occur when signaling is completely wasteful.

We focus on a simple example with two agents on each side, and with constant marginal external benefits. From $r(2,2) = 1/9$ and Theorem 1, we know that the side-symmetric, strictly separating equilibrium exists if and only if $d_M d_W < 9$. Moreover, the equilibrium strategies satisfy

$$b_M(m) = \gamma_M a_M(m) + d_M F(m) \Delta_{W,1}^{b_W} \tag{14}$$

and

$$b_W(w) = \gamma_W a_W(w) + d_W G(w) \Delta^{b_M}_{M,1}, \tag{15}$$

where $\Delta^{b_W}_W$ and $\Delta^{b_M}_M$ are explicitly given by

$$\Delta^{b_M}_M = \begin{pmatrix} 1 & -\frac{d_M d_W}{9-d_M d_W} \\ 0 & \frac{9}{9-d_M d_W} \end{pmatrix} \left(\gamma_M \Delta^{a_M}_M + d_M \gamma_W \begin{pmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \Delta^{a_W}_W \right)$$
$$\Delta^{b_W}_W = \begin{pmatrix} 1 & -\frac{d_M d_W}{9-d_M d_W} \\ 0 & \frac{9}{9-d_M d_W} \end{pmatrix} \left(\gamma_W \Delta^{a_W}_W + d_W \gamma_M \begin{pmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \Delta^{a_M}_M \right).$$

In particular, the second-row entries (with index 1), which occur in (14) and (15) are

$$\Delta^{b_M}_{M,1} = \frac{9}{9-d_M d_W} \left(\gamma_M \Delta^{a_M}_{M,1} + d_M \gamma_W \frac{1}{3} \Delta^{a_W}_{W,1} \right)$$

and

$$\Delta^{b_W}_{W,1} = \frac{9}{9-d_M d_W} \left(\gamma_W \Delta^{a_W}_{W,1} + d_W \gamma_M \frac{1}{3} \Delta^{a_M}_{M,1} \right). \tag{16}$$

To explicitly pin down the equilibrium strategies, we thus only need to identify the terms $\Delta^{a_M}_{M,1}$ and $\Delta^{a_W}_{W,1}$. From Lemma 4 in the Appendix we obtain

$$\Delta^{a_M}_{M,1} = \frac{1}{3} E[M_{2:3}] E[W_{2:2} - W_{1:2}] \quad \text{and} \quad \Delta^{a_W}_{W,1} = \frac{1}{3} E[W_{2:3}] E[M_{2:2} - M_{1:2}]. \tag{17}$$

For the remainder of this example, we assume $d = d_M = d_W$, $F = G$ and $\underline{m} = 0$.[27] We give conditions such that $E[b_M(M_{1:2})]$ is strictly increasing in γ_M. Since $dE[b_M(M_{1:2})]$ equals the expected utility of a woman with type zero (see Lemma 3), this implies, by continuity of the distribution and of the bid functions, that there is a positive measure of (low) women types whose expected utility decreases when women's match share $\gamma_W = 1 - \gamma_M$ increases.[28]

At first sight, this may seem counter-intuitive. The argument behind it is that low types get most of their utility from their partner's investments. In contrast to their prospective partners, these low types "know" already that the matching outcome will be low. But, the direct benefits of the partner's investment – who expects a better match than he or she gets in the end – are attractive in comparison. As an increase in intra-household bargaining power for one market side leads to a decrease in the potential partners' investments, low types suffer from their empowerment since their expected

[27]It is not difficult to construct similar examples with asymmetric external benefits and distributions, and also with $\underline{m} > 0$, provided that $\overline{m}$ is sufficiently large.

[28]By symmetry, an analogous result applies for low type men.

utility decreases.

Here is the formal argument. By Lemma 4 in the Appendix , $E[F(M_{1:2})] = \theta_{2,01} = \frac{1}{3}$ and $E[a_M(M_{1:2})] = \frac{1}{3}E[M_{1:3}]E[W_{2:2} - W_{1:2}]$. From (14), (16) and (17) we obtain that

$$E[b_M(M_{1:2})] = \left(\gamma_M E[M_{1:3}] + d\,\frac{3\gamma_W + d\,\gamma_M}{9-d^2}\,E[M_{2:3}]\right)\frac{E[M_{2:2} - M_{1:2}]}{3}.$$

Substituteing $\gamma_W = 1 - \gamma_M$ we obtain

$$E[b_M(M_{1:2})] = \left(\frac{3d}{9-d^2}E[M_{2:3}] + \gamma_M\left(E[M_{1:3}] - \frac{d}{3+d}E[M_{2:3}]\right)\right)\frac{E[M_{2:2} - M_{1:2}]}{3}.$$

Consequently, $dE[b_M(M_{1:2})]$ is strictly increasing in γ_M if and only if $(\frac{3}{d}+1)E[M_{1:3}] > E[M_{2:3}]$. In particular, Lemma 5 in the Appendix shows that this is the case if investments are partially wasteful ($0 < d < 1$) and if the function $m - \frac{1-F(m)}{f(m)}$ is increasing. The assumption of increasing virtual valuations implies $4E[M_{1:3}] \geq E[M_{2:3}]$.

3.1.3 Bounds on Under-Investment when Investments are Productive

We may use Theorem 1 to derive some interesting bounds on under-investment for situations with investments that are productive. We say that investments are *productive* if the following condition holds:

$$\delta_M'(0)\delta_W'(0) > 1 \text{ and } \lim_{b\to\infty}\delta_M'(b)\lim_{b\to\infty}\delta_W'(b) < 1. \tag{18}$$

Under condition (18), the pairs of investments (β_M, β_W) that are Pareto efficient and individually rational for a given pair of agents are characterized by the conditions

$$\delta_M'(\beta_W)\delta_W'(\beta_M) = 1,\ \delta_M(\beta_W) - \beta_M \geq 0 \text{ and } \delta_W(\beta_M) - \beta_W \geq 0.$$

All these pairs involve investments strictly above the privately optimal levels $\beta_M = 0$ and $\beta_W = 0$. Pairs of investments satisfying $\delta_M'(\beta_W)\delta_W'(\beta_M) > 1$ are inefficiently low then,[29] while investments $(\beta_M, \beta_W) \in R^2_{++}$ for which $\delta_M'(\beta_W)\delta_W'(\beta_M) < 1$ correspond to over-investment. Moreover, the level sets

$$L_c := \{(\beta_M, \beta_W) \in R^2_{++} | \delta_M'(\beta_W)\delta_W'(\beta_M) = c\}$$

are closer to L_1 the closer c is to 1.

[29] There would be a way of marginally increasing both agents' investments that yields a Pareto improvement for the pair.

If investments are productive, the fact that an agent's investment benefits his/her partner creates a standard incentive for under-investment (the "hold-up" problem). This problem is at least partly mitigated by the competition on each side of the market for the external benefits offered by the other side. Moreover, the "exogenous" signaling motive (the competition for partners with higher *types*) provides additional incentives. However, our results imply that if n and k are such that $r(n,k)\delta_M'(0)\delta_W'(0) < 1$, the feedback effect is not strong enough for investments to exceed a bounded multiple of the pure signaling investments. In particular, if signaling incentives are small, the competition among n men and k women does not suffice to avoid extreme under-investment. We formalize this finding in Corollary 1, where n, k, δ_M and δ_W are fixed, and where the conditions on the supports of types are necessary and sufficient for signaling incentives to become arbitrarily small, independently of the exact form of F and G.[30]

Corollary 1. *Assume that $r(n,k)\delta_M'(0)\delta_W'(0) < 1$. If $n = k$, $\overline{m}(\overline{w} - \underline{w}) \to 0$ and $(\overline{m} - \underline{m})\overline{w} \to 0$, or if $n > k$ and $\overline{m}\overline{w} \to 0$, then $b_M(\overline{m}) \to 0$ and $b_W(\overline{w}) \to 0$ for any side-symmetric, strictly separating equilibrium.*

By contrast, if n and k are such that $r(n,k)\delta_M'(0)\delta_W'(0) > 1$, the feedback effect is too strong for investments to get stuck at extremely low levels.

Corollary 2. *In any side-symmetric, strictly separating equilibrium, it holds that*

$$r(n,k)\delta_M'(b_W(\overline{w}))\delta_W'(b_M(\overline{m})) < 1.$$

Thus, if $r(n,k)\delta_M'(0)\delta_W'(0) > 1$, we obtain a quantitative bound on under-investment: even with arbitrarily small signaling concerns, it is guaranteed that the investments made by the highest types, $(b_M(\overline{m}), b_W(\overline{w}))$, lie above the level set $L_{1/r(n,k)}$.

The bounds of Corollaries 1 and 2 are interesting for understanding the extent to which the feedback effect can rule out "extreme" under-investment in small markets. They do not provide novel information about equilibrium behavior in the limit of very large markets however (see the explanations in Section 4).

4 Large Contests with Partially Wasteful or TU Investments

Theorem 1 permits some new insights into the relationship between finite and continuum models of matching contests: it allows us to obtain precise characterizations of

[30]Thus, $\Delta_M^{a_M}$ and $\Delta_W^{a_W}$ converge to zero.

the limit properties of equilibrium utilities in large matching contests if investments are either partially wasteful (i.e., $\delta_M'(0)\delta_W'(0) < 1$, see Section 3.1.1) or correspond to monetary transfers under quasi-linear utility, i.e., $d_M = d_W = 1$. We refer to the latter case as the *TU investment* case.[31] A thorough analysis of asymptotic behavior if investments are productive (i.e., $\delta_M'(0)\delta_W'(0) > 1$, see Section 3.1.3) is way beyond the scope of the present study.[32] However, it is clear from Peters (2011) that, in this case almost all types must weakly over-invest in the limit of very large markets even if signaling incentives become arbitrarily small.[33]

For the following analysis, we have to assume that matching with a partner who does not invest and has the lowest possible type is not better than staying unmatched, formalized here by setting $\underline{w} = 0$.[34] To simplify notation, we assume $\underline{w} = 0$ also for balanced contests, and let $\underline{m} = 0$, but these two assumptions do not matter for any of the results.

Condition 1. $\underline{w} = \underline{m} = 0$.

The Continuum Model

We consider here the limit two-sided matching contest with a continuum of agents. The distribution of women is G, the distribution of men is $F/(1-r)$ for some $r \in [0,1)$,[35] and types are private information. Let m_r denote the r-th quantile of F (i.e., $F(m_r) = r$), and let $F_r(m) = (F(m) - F(m_r))/(1-r)$ for $m \geq m_r$. The positively assortative matching is now described by the matching function

$$\psi_r(m) = \begin{cases} 0 & \text{if } m < m_r \\ G^{-1}(F_r(m)) & \text{if } m \geq m_r, \end{cases}$$

which is strictly increasing on $[m_r, \overline{m}]$, while types below m_r stay unmatched. We let ϕ_r denote its inverse, defined on $[0, \overline{w}]$.

If all other agents invest according to non-decreasing functions $b_M : [0, \overline{m}] \to \mathbb{R}_+$ and $b_W : [0, \overline{w}] \to \mathbb{R}_+$ that are strictly increasing on $[m_r, \overline{m}]$ and $[0, \overline{w}]$, then a man who invests $b_M(s)$ knows that he will be matched to a woman with type $\psi_r(s)$ who makes

[31] Note that in this case, all possible (β_M, β_W) are Pareto efficient for a given pair of agents.

[32] The fact that investments "blow up" in a way that is unbounded by the size of the pure signaling incentives in this case (see Section 3.1.3) combined with the non-linearity of the benefit functions creates problems that cannot be addressed with the paper's techniques.

[33] One could also mimic the qualitative arguments in Peters (2011) to give conditions on the shape of δ_M and δ_W that ensure strict over-investment in very large, unbalanced markets.

[34] If k is large then $E[W_{1:k}] \approx \underline{w}$, so $\underline{w} = 0$ implies that a match with the lowest-ranked woman is not a "large prize" for exogenous reasons (i.e., even if the woman does not invest).

[35] Thus, the total mass of men is $1/(1-r) \geq 1$.

an investment $b_W(\psi_r(s))$. Similarly, a woman who invests $b_W(s)$ is matched to a man with type $\phi_r(s)$ whose investment is $b_M(\phi_r(s))$. Thus, returns to investments are here certain, a significant difference to the finite case.

Equilibrium strategies must clearly satisfy $b_M(m) = 0$ for $m < m_r$ (as these types stay unmatched for sure). Moreover, if $b_W(0) = 0$ and b_M and b_W are continuous (as in the equilibrium for the partially wasteful case, Theorem 3i below), issues related to the question of how to define returns for investments outside of the ranges $b_M([0,\overline{m}])$ and $b_W([0,\overline{w}])$ do not arise.[36]

Theorem 3. *Assume that Condition 1 holds.*

i) If $\delta'_M(0)\delta'_W(0) < 1$, then the continuum model admits a side-symmetric equilibrium (b_M, b_W) such that b_M and b_W are continuous, $b_W(0) = 0$, and b_M and b_W are strictly increasing and continuously differentiable on $[m_r,\overline{m}]$ and $[0,\overline{w}]$, respectively. There is a unique equilibrium with these properties. Agents' equilibrium utilities are given by

$$u_M^{(r)}(m) = \gamma_M \int_0^m \psi_r(s)ds \quad \text{and} \quad u_W^{(r)}(w) = \gamma_W \int_0^w \phi_r(s)ds.$$

ii) If $\delta_M(\beta_W) = \beta_W$ and $\delta_W(\beta_M) = \beta_M$, the continuum model does not admit an equilibrium that implements the positive assortative matching.

Thus, with partially wasteful investments, assortative matching can arise in equilibrium. With TU investments (case ii), the continuum model does not admit such an equilibrium.

Limit Characterizations

We now return to our main focus on finite markets. Given the separable form of the utility functions, we can use the standard payoff equivalence result for Bayesian incentive compatible mechanisms in order to represent interim expected utilities. For a side-symmetric, strictly separating equilibrium, we let $U_M(m)$ denote the expected utility for a man with type m, and $U_W(w)$ denote the expected utility for a woman with type w. We write $\Psi(m) \in [0,\overline{w}]$ for type m's expected type of partner and $\Phi(w) \in [\underline{m},\overline{m}]$ for

[36] In contrast, the definition of off-equilibrium payoffs plays an important role in continuum models with productive investments, and many different outcomes can potentially be supported as equilibria, depending on the choice of definition. See Peters (2011) for a detailed discussion of this phenomenon.

type w's expected type of partner. Thus,

$$\Psi(m) = \begin{cases} E[W_{1:n}] + \mathscr{F}(m) \cdot \Delta_W^{I_1} & \text{if } n = k \\ \hat{\mathscr{F}}(m) \cdot \Delta_W^{I_1} & \text{if } n > k. \end{cases}$$

$$\Phi(w) = E[M_{n-k+1:n}] + \mathscr{G}(w) \cdot \hat{\Delta}_M^{I_1}.$$

Lemma 3. *Agents' interim expected utilities satisfy:*

$$U_M(m) = \begin{cases} E[\delta_M(b_W(W_{1:n}))] + \gamma_M \underline{m} E[W_{1:n}] + \gamma_M \int_{\underline{m}}^m \Psi(s) ds & \textit{if } n = k \\ \gamma_M \int_{\underline{m}}^m \Psi(s) ds & \textit{if } n > k. \end{cases} \tag{19}$$

$$U_W(w) = E[\delta_W(b_M(M_{n-k+1:n}))] + \gamma_W \underline{w} E[M_{n-k+1:n}] + \gamma_W \int_{\underline{w}}^w \Phi(s) ds. \tag{20}$$

We now fix all characteristics of the environment other than the number of agents. That is, F, G, γ_M, γ_W, δ_M and δ_W are arbitrary but fixed throughout Section 4. We use here superscripts to highlight the dependence of strategies, utilities and expected match partners on n and k, writing $b_M^{(n,k)}$, $b_W^{(n,k)}$, $a_M^{(n,k)}$, $a_W^{(n,k)}$, $\Psi^{(n,k)}$, $\Phi^{(n,k)}$, $U_M^{(n,k)}$ and $U_W^{(n,k)}$. We first show that agents' expected utilities in large matching contests with partially wasteful investments converge to the equilibrium utilities for the continuum model:

Theorem 4. *Assume that Condition 1 is satisfied and that $\delta_M'(0)\delta_W'(0) < 1$. Then, for all $m \in [0, \overline{m}]$ and $w \in [0, \overline{w}]$:*

$$\lim_{k\to\infty, \frac{k}{n_k}\to 1-r} U_M^{(n_k,k)}(m) = u_M^{(r)}(m) \quad \textit{and} \quad \lim_{k\to\infty, \frac{k}{n_k}\to 1-r} U_W^{(n_k,k)}(w) = u_W^{(r)}(w).$$

In any finite market, a type's expected utility exceeds his or her *information rent*, $\gamma_M \int_0^m \Psi^{(n,k)}(s) ds$ or $\gamma_W \int_0^w \Phi^{(n,k)}(s) ds$ by a constant equal to the expected benefit from the guaranteed partner's investment, i.e., by $E[\delta_M(b_W^{(n,n)}(W_{1:n}))]$ or 0 for men, and by $E[\delta_W(b_M^{(n,k)}(M_{n-k+1:n}))]$ for women.[37] Theorem 4 shows that this extra utility converges to zero if $\delta_M'(0)\delta_W'(0) < 1$, and also that information rents converge to their continuum counterparts.

Even though signals are only partially wasteful, Theorem 4 implies that, asymptotically, the *entire* difference between total match surplus

$$S^{(r)} := \int_{m_r}^{\overline{m}} m \psi_r(m) \frac{f(m)}{1-r} dm$$

[37]Note that the terms $\gamma_M \underline{m} E[W_{1:n}]$ and $\gamma_W \underline{w} E[M_{n-k+1:n}]$ (the type-dependent component of the lowest type's utility from his/her guaranteed match) in (19) and (20) vanish under Condition 1.

and aggregate information rents

$$\gamma_M R_M^{(r)} + \gamma_W R_W^{(r)},$$

where

$$R_M^{(r)} = \int_{m_r}^{\overline{m}} \left(\int_0^m \psi_r(s) ds \right) \frac{f(m)}{1-r} dm \quad \text{and} \quad R_W^{(r)} = \int_0^{\overline{w}} \left(\int_0^w \phi_r(s) ds \right) g(w) dw,$$

gets dissipated, in accordance with the prediction of Theorem 3 (i). Note that the dissipation rate is always the same, no matter what the benefit functions δ_M and δ_W are (as long as the condition $\delta_M'(0)\delta_W'(0) < 1$ is satisfied). For example, if $\gamma_M = \gamma_W = \frac{1}{2}$, exactly half the available surplus is dissipated in the limit, exactly as in the HMS model where signals are completely wasteful and bring no benefit to the other side.

With TU investments, the continuum model does not admit a separating equilibrium (part (ii) of Theorem 3) since investments grow out of bounds. In the finite model, equilibrium strategies become very steep as n and k grow, but a unique side-symmetric, strictly separating equilibrium always exists (by Theorem 1 and $r(n,k) < 1$)!

Interestingly, we are able to precisely characterize the limit behavior of the corresponding equilibrium utilities. This amounts here to understand how the expected investments of the types $M_{n-k+1:n}$ and $W_{1:n}$ behave as the market grows.

Note first that ex-post budget balance holds here for each matched pair. In particular, in any *balanced* market the sum of all agents' ex-ante expected utilities must be equal to the ex-ante expected total match surplus. That is, for all n:

$$\begin{aligned} E\left[\frac{1}{n}\sum_{i=1}^{n} M_{i:n}W_{i:n}\right] &= \int_0^{\overline{m}} U_M^{(n,n)}(m) f(m) dm + \int_0^{\overline{w}} U_W^{(n,n)}(w) g(w) dw \\ &= E[b_W^{(n,n)}(W_{1:n})] + \gamma_M \int_0^{\overline{m}} \int_0^m \Psi^{(n,n)}(s) ds f(m) dm \\ &\quad + E[b_M^{(n,n)}(M_{1:n})] + \gamma_W \int_0^{\overline{w}} \int_0^w \Phi^{(n,n)}(s) ds g(w) dw. \end{aligned}$$

As $n \to \infty$, the integral terms converge to $\gamma_M R_M^{(0)}$ and $\gamma_W R_W^{(0)}$ (by Lemma 7 in the Appendix and the Dominated Convergence Theorem), and the left hand side converges to $S^{(0)}$ (by the Law of Large Numbers for empirical distributions). Thus,

$$S^{(0)} = \lim_{n\to\infty} (E[b_M^{(n,n)}(M_{1:n})] + E[b_W^{(n,n)}(W_{1:n})]) + \gamma_M R_M^{(0)} + \gamma_W R_W^{(0)}.$$

Because $R_M^{(0)}$ and $R_W^{(0)}$ are in fact the aggregated core (or stable) payoffs for men and women in the continuum model, we must have

$$S^{(0)} = R_M^{(0)} + R_W^{(0)}.$$

Thus, we obtain

$$\lim_{n\to\infty}(E[b_M^{(n,n)}(M_{1:n})] + E[b_W^{(n,n)}(W_{1:n})]) = \gamma_W R_M^{(0)} + \gamma_M R_W^{(0)} > 0.$$

We show that the *difference* between the expected investments of $M_{1:n}$ and $W_{1:n}$ converges to 0, i.e., $\lim_{n\to\infty}(E[b_M^{(n,n)}(M_{1:n})] - E[b_W^{(n,n)}(W_{1:n})]) = 0$, leading to our limit characterization of equilibrium utilities in the TU case.

Theorem 5. *Assume that $\delta_M(\beta_W) = \beta_W$, $\delta_W(\beta_M) = \beta_M$ and that Condition 1 is satisfied.*

i) In large balanced markets, the (per capita) difference between aggregate surplus and aggregate information rents is shared approximately fifty-fifty between men and women, regardless of F, G and γ_M: for all $m \in [0, \overline{m}]$ and $w \in [0, \overline{w}]$,

$$\lim_{n\to\infty} U_M^{(n,n)}(m) = \frac{\gamma_W R_M^{(0)} + \gamma_M R_W^{(0)}}{2} + \gamma_M \int_0^m \psi_0(s)ds,$$
$$\lim_{n\to\infty} U_W^{(n,n)}(w) = \frac{\gamma_W R_M^{(0)} + \gamma_M R_W^{(0)}}{2} + \gamma_W \int_0^w \phi_0(s)ds.$$

ii) In large, unbalanced markets, each agent on the short side obtains a fraction close to 1 of the per capita difference between aggregate surplus and aggregate information rents (on top of her information rent): for all $m \in [0, \overline{m}]$ and $w \in [0, \overline{w}]$,

$$\lim_{k\to\infty, k<n_k, \frac{k}{n_k}\to 1-r} U_M^{(n_k,k)}(m) = \gamma_M \int_0^m \psi_r(s)ds,$$
$$\lim_{k\to\infty, k<n_k, \frac{k}{n_k}\to 1-r} U_W^{(n_k,k)}(w) = \gamma_W R_M^{(r)} + \gamma_M R_W^{(r)} + \gamma_W \int_0^w \phi_r(s)ds.$$

The proof of Theorem 5 (i) relies on some subtle properties of the matrix Θ_n. The unbalanced case in Theorem 5 (ii) is much simpler because only agents on the short side obtain utility in excess of information rents, and because only a vanishingly small fraction of total surplus can get dissipated (the investments made by agents failing to match) in large markets.

5 Conclusion

We have provided the first quantitative analysis of the feedback effect shaping Bayes-Nash equilibrium investments in two-sided matching contests when the investments create external benefits, and studied how the effect interacts with agents' signaling incentives. We have identified the Perron root of a particular matrix of moments of order statistics as the key measure of the intensity of the feedback cycle. This characterization has allowed us to obtain detailed information about how the strength of the feedback effect and the resulting amplification of agents' pure signaling investments depend on market size.

We have illustrated our results by highlighting some surprising consequences for the comparative statics of total equilibrium signaling, and by providing novel, quantitative insights into how competition alleviates the hold-up problem in small markets with productive investments. The complex, interdependent nature of equilibrium behavior also gives rise to other intricate effects. For example, in small markets, raising the surplus share of one side of the market may impact the expected utility of agents on that side in very different ways. Specifically, the analysis documents that an increase in bargaining power on the own market side can harm some agents. Especially those with little to offer may suffer.

Finally, our results have also allowed us to shed new light on the relationship between finite and continuum models of matching contests by deriving the exact asymptotic behavior of equilibrium utilities, as the numbers of men and women go to infinity, in environments with partially wasteful or TU investments.

Appendix

Proof of Lemma 1. If b_W is an equilibrium strategy, it is immediate from (6), (7) and the definitions of T and S that $\Delta_W^{\delta_M \circ b_W}$ is a fixed point of $S \circ \hat{T}$. Moreover, the mapping is one-to-one (if $\iota(b_W^1) = \iota(b_W^2)$ for two equilibrium strategies b_W^1 and b_W^2, then (4) and (5) imply $b_W^1 = b_W^2$) and onto: if y^* is a fixed point of $S \circ \hat{T}$, then $b_M(m) := \gamma_M a_M(m) + \hat{\mathscr{F}}(m) \cdot y^*$ and $b_W(w) := \gamma_W a_W(w) + \mathscr{G}(w) \cdot \hat{\Delta}_M^{\delta_W \circ b_M}$ constitute equilibrium strategies satisfying $\Delta_W^{\delta_M \circ b_W} = y^*$. $\square$

Proof of Lemma 2. i) $\theta_{n,i0} = 0$ is obvious. For $j \geq 1$, we first rewrite $\theta_{n,ij}$ using integration by parts:[38]

$$\theta_{n,ij} = E[F_{j:n-1}(M_{i+1:n}) - F_{j:n-1}(M_{i:n})] = \int_{\underline{m}}^{\overline{m}} F_{j:n-1}(m)(f_{i+1:n}(m) - f_{i:n}(m))dm$$
$$= \int_{\underline{m}}^{\bar{m}} (F_{i:n}(m) - F_{i+1:n}(m))f_{j:n-1}(m)dm = E[F_{i:n}(M_{j:n-1}) - F_{i+1:n}(M_{j:n-1})].$$

Using the identities (1) and (2), it follows that

$$\theta_{n,ij} = E[F_{i:n}(M_{j:n-1}) - F_{i+1:n}(M_{j:n-1})] = E\left[\binom{n}{i} F^i(M_{j:n-1})(1-F(M_{j:n-1}))^{n-i}\right]$$
$$= \binom{n}{i}(n-1)\binom{n-2}{j-1}\int_{\underline{m}}^{\overline{m}} F^{i+j-1}(m)(1-F(m))^{2n-1-i-j} f(m)dm$$
$$= \frac{n-1}{2n-1}\frac{\binom{n}{i}\binom{n-2}{j-1}}{\binom{2n-2}{i+j-1}}\int_{\underline{m}}^{\overline{m}} f_{i+j:2n-1}(m)dm = \frac{n-1}{2n-1}\frac{\binom{n}{i}\binom{n-2}{j-1}}{\binom{2n-2}{i+j-1}}.$$

ii) *Case n=2*: The formula from (i) yields $\theta_{2,01} = \theta_{2,11} = \frac{1}{3}$. Thus, $3\Theta_2$ is stochastic.

Case $n > 2$: The entries of the i-th row of Θ_n can be reinterpreted in a way that allows computing the row sum as a telescoping sum. Indeed, for $j = 2, ..., n-2$:

$$E[F_{j-1:n-2}(M_{i+1:n+1}) - F_{j:n-2}(M_{i+1:n+1})]$$
$$= \int_{\underline{m}}^{\overline{m}} \binom{n-2}{j-1} F^{j-1}(m)(1-F(m))^{n-1-j} f_{i+1:n+1}(m)dm$$
$$= \binom{n-2}{j-1}(n+1)\binom{n}{i}\int_{\underline{m}}^{\overline{m}} F^{i+j-1}(m)(1-F(m))^{2n-1-i-j} f(m)dm$$
$$= \frac{n+1}{2n-1}\frac{\binom{n}{i}\binom{n-2}{j-1}}{\binom{2n-2}{i+j-1}} = \frac{n+1}{n-1}\,\theta_{n,ij}.$$

Similarly, for $j = n-1$ we find:

$$E[F_{n-2:n-2}(M_{i+1:n+1})] = \int_{\underline{m}}^{\overline{m}} \binom{n-2}{n-2} F^{n-2}(m) f_{i+1:n+1}(m)dm$$
$$= \binom{n-2}{n-2}(n+1)\binom{n}{i}\int_{\underline{m}}^{\overline{m}} F^{i+(n-1)-1}(m)(1-F(m))^{2n-1-i-(n-1)} f(m)dm$$
$$= \frac{n+1}{2n-1}\frac{\binom{n}{i}\binom{n-2}{n-2}}{\binom{2n-2}{i+(n-1)-1}} = \frac{n+1}{n-1}\,\theta_{n,i(n-1)}.$$

Finally, for $j = 1$,

$$E[1 - F_{1:n-2}(M_{i+1:n+1})] = \frac{n+1}{n-1}\,\theta_{n,i1}.$$

[38] Observe that the case $i = 0$ (so that $F_{0:n} \equiv 1$ on $[\underline{m}, \overline{m}]$) is covered by the argument.

Summing up all terms (including $\theta_{n,i0}=0$) yields $\frac{n+1}{n-1}\sum_{j=0}^{n-1}\theta_{n,ij}=1$. $\square$

Proof of Theorem 1. If $d_M=0$, the unique fixed point of $S\circ\hat{T}$ is 0, and (11) and (12) follow immediately from (9) and (10). If $d_M>0$, the fixed point equation becomes

$$\Delta_W^{b_W}=\gamma_W\Delta_W^{a_W}+d_W\Theta_k(\gamma_M\hat{\Delta}_M^{a_M}+d_M\hat{\Theta}_{n,k}\Delta_W^{b_W}),$$

or

$$(I_k-d_Md_W\Theta_k\hat{\Theta}_{n,k})\Delta_W^{b_W}=\gamma_W\Delta_W^{a_W}+d_W\gamma_M\Theta_k\hat{\Delta}_M^{a_M}.^{39} \tag{21}$$

Case $r(n,k)d_Md_W<1$: $\rho(d_Md_W\Theta_k\hat{\Theta}_{n,k})=r(n,k)d_Md_W<1$ implies in particular that $I_k-d_Md_W\Theta_k\hat{\Theta}_{n,k}$ is invertible, so that (21) yields (12). We must also show $\Delta_W^{b_W}\geq 0$. As $\rho(d_Md_W\Theta_k\hat{\Theta}_{n,k})<1$, $(I_k-d_Md_W\Theta_k\hat{\Theta}_{n,k})^{-1}$ is given by the convergent series $\sum_{l=0}^{\infty}(d_Md_W\Theta_k\hat{\Theta}_{n,k})^l$ (Theorem 5.6.15 and Corollary 5.6.16 in Horn and Johnson 2013). Thus, $(I_k-d_Md_W\Theta_k\hat{\Theta}_{n,k})^{-1}\geq 0$. Moreover, $\Delta_W^{a_W}>0$ because a_W is strictly increasing (for $j\geq 1$, $w\mathcal{G}_j(w)-\int_{\underline{w}}^{w}\mathcal{G}_j(s)ds$ is strictly increasing, and $\hat{\Delta}_M^{I_1}>0$). Similarly, $\hat{\Delta}_M^{a_M}>0$. As $\Theta_k\geq 0$, it follows that $\gamma_W\Delta_W^{a_W}+d_W\gamma_M\Theta_k\hat{\Delta}_M^{a_M}>0$. Thus, $\Delta_W^{b_W}\geq 0$ (in fact, $\Delta_W^{b_W}>0$, because the diagonal entries of $(I_k-d_Md_W\Theta_k\hat{\Theta}_{n,k})^{-1}$ are positive). Finally, the explicit form of $\hat{\Delta}_M^{b_M}$, stated in (11), follows from plugging (10) into (9).

Case $r(n,k)d_Md_W\geq 1$: let $z:=\gamma_W\Delta_W^{a_W}+d_W\gamma_M\Theta_k\hat{\Delta}_M^{a_M}$. We have already shown that $z>0$. We have to show that the linear system of equations $(I_k-d_Md_W\Theta_k\hat{\Theta}_{n,k})v=z$ has no solution $v\in\mathbb{R}_+^k$. This follows from Farkas' Lemma and from the fact that the non-negative and non-zero matrix $\Theta_k\hat{\Theta}_{n,k}$, has a non-negative, non-zero left eigenvector $w(n,k)$ associated with $r(n,k)$, i.e., $w(n,k)^T\Theta_k\hat{\Theta}_{n,k}=r(n,k)w(n,k)^T$ (see Theorem 8.3.1 in Horn and Johnson 2013). It follows that

$$w(n,k)^T(I_k-d_Md_W\Theta_k\hat{\Theta}_{n,k})=w(n,k)^T(1-r(n,k)d_Md_W)\leq 0 \text{ and } w(n,k)^Tz>0.$$

Thus, by Farkas' Lemma, $(I_k-d_Md_W\Theta_k\hat{\Theta}_{n,k})v=z$ has no solution $v\geq 0$, which concludes the proof for the case $r(n,k)d_Md_W\geq 1$. $\square$

Proof of Theorem 2. Case $r(n,k)(\lim_{b\to\infty}\delta_M'(b))(\lim_{b\to\infty}\delta_W'(b))<1$: We must show that $S\circ\hat{T}$ has a fixed point. We prove this for the case $n>k$ first. Let C_M and C_W be two constants satisfying $\lim_{b\to\infty}\delta_M'(b)<C_M$, $\lim_{b\to\infty}\delta_W'(b)<C_W$ and $r(n,k)C_WC_M<1$. We show below that for all $y\in\mathbb{R}_+^k$

$$S(\hat{T}(y))\leq C_MC_W\Theta_k\hat{\Theta}_{n,k}y+R(y), \tag{22}$$

[39] We have derived the equation directly from (9) and (10). This is equivalent to writing down the fixed point equation $S\circ\hat{T}(y)=y$ for $y=d_M\Delta_W^{b_W}$ and then dividing this equation by the constant $d_M>0$.

for some $R : \mathbb{R}^k_+ \to \mathbb{R}^k_+$ satisfying $R(y) = o(||y||)$ as $||y|| \to \infty$ for any norm $||\cdot||$ on $\mathbb{R}^k$ (here $o(\cdot)$ is the usual "little o" Landau symbol, and the inequality (22) holds of course coordinate-wise). Invoking some facts from matrix analysis, this implies the existence of a compact and convex set that is mapped into itself by $S \circ \hat{T}$, so that, by Brouwer's Theorem, $S \circ \hat{T}$ has a fixed point. To prove the existence of such a set, we show first that $C_M C_W \Theta_k \hat{\Theta}_{n,k} > 0$ and $\rho(C_M C_W \Theta_k \hat{\Theta}_{n,k}) = C_M C_W r(n,k) < 1$ imply that there is a *monotone* norm $\|\cdot\|$ on $\mathbb{R}^k$ (i.e., $|y_i| \leq |\tilde{y}_i|$ for all i implies $\|y\| \leq \|\tilde{y}\|$) such that $C_M C_W \Theta_k \hat{\Theta}_{n,k}$ is a contraction with respect to $\|\cdot\|$.[40] Indeed, for an arbitrary positive $k \times k$ matrix $A > 0$, let $x > 0$ be its *Perron (right) eigenvector*, i.e., $Ax = \rho(A)x$ (see Theorem 8.2.8 in Horn and Johnson 2013) and define D as the diagonal matrix $D = \text{diag}(x_1, ..., x_k)$. By Theorem 5.6.7 of Horn and Johnson (2013), $|||B|||_{D^{-1}} := |||D^{-1}BD|||_\infty$ defines a matrix norm that is induced by the vector norm $\|\cdot\|_{D^{-1}}$ defined via $\|y\|_{D^{-1}} := \|D^{-1}y\|_\infty$. That is, for an arbitrary matrix $B \in \mathbb{R}^{k\times k}$, $|||B|||_{D^{-1}} = \max_{y\neq 0} \frac{\|By\|_{D^{-1}}}{\|y\|_{D^{-1}}}$. A straightforward calculation shows that all row sums of $D^{-1}AD$ are equal to $\rho(A)$ (see exercise 8.2.P12 in Horn and Johnson 2013). Using that $|||\cdot|||_\infty$ is the "row sum norm", i.e., for any matrix B, $|||B|||_\infty = \max_{0\leq i\leq k-1} \sum_{j=0}^{k-1} |b_{ij}|$ (Example 5.6.5 in Horn and Johnson 2013), it follows that $|||A|||_{D^{-1}} = |||D^{-1}AD|||_\infty = \rho(A)$. Moreover, the vector norm $\|\cdot\|_{D^{-1}}$ is clearly monotone. Applying these findings for $A = C_M C_W \Theta_k \hat{\Theta}_{n,k}$, we obtain from (22) (note that the first inequality uses the monotonicity of the norm):

$$\begin{aligned}\|S(\hat{T}(y))\|_{D^{-1}} &\leq \|C_M C_W \Theta_k \hat{\Theta}_{n,k} y + R(y)\|_{D^{-1}} \leq \|C_M C_W \Theta_k \hat{\Theta}_{n,k} y\|_{D^{-1}} + \|R(y)\|_{D^{-1}} \\ &\leq |||C_M C_W \Theta_k \hat{\Theta}_{n,k}|||_{D^{-1}} \|y\|_{D^{-1}} + \|R(y)\|_{D^{-1}} = \rho(C_M C_W \Theta_k \hat{\Theta}_{n,k}) \|y\|_{D^{-1}} + o(\|y\|_{D^{-1}}).\end{aligned} \tag{23}$$

For any $K > 0$, let $\overline{B}_K(0) = \{y \in \mathbb{R}^k : \|y\|_{D^{-1}} \leq K\}$ (the closed ball of radius K for the norm $||\cdot||_{D^{-1}}$). Clearly, $\overline{B}_K(0)$ is compact and convex (see also Theorem 5.5.8 in Horn and Johnson 2013). As $\rho(C_M C_W \Theta_k \hat{\Theta}_{n,k}) < 1$, (23) implies that there is some $K_1 < \infty$ such that for all $y \in \mathbb{R}^k_+$ with $||y||_{D^{-1}} \geq K_1$, $||S(\hat{T}(y))||_{D^{-1}} \leq ||y||_{D^{-1}}$. Moreover, $S(\hat{T}(\overline{B}_{K_1}(0) \cap \mathbb{R}^k_+))$ is compact ($\overline{B}_{K_1}(0) \cap \mathbb{R}^k_+$ is compact and $S \circ \hat{T}$ is continuous). Taken together, these two observations imply that there is some K_2 (possibly greater than K_1), such that $S(\hat{T}(\overline{B}_{K_2}(0) \cap \mathbb{R}^k_+)) \subset \overline{B}_{K_2}(0) \cap \mathbb{R}^k_+$, i.e., the continuous mapping $S \circ \hat{T}$ maps the compact and convex set $\overline{B}_{K_2}(0) \cap \mathbb{R}^k_+$ into itself. Thus, it has a fixed point.

We still have to show (22). Note first that by the definition of C_M and C_W, there are constants $0 < b^1 < b^2 < \infty$ such that for all $b \geq b^1$, $\delta'_M(b) \leq C_M$ and $\delta'_W(b) \leq C_W$, and

[40] Note that for many "naive" guesses of monotone norms, $C_M C_W \Theta_k \hat{\Theta}_{n,k}$ need not be a contraction. E.g., for a positive matrix A for which not all row sums are equal, we generally have the strict inequality $\rho(A) < |||A|||_\infty$. Thus, even if $\rho(A) < 1$ there may be vectors for which $\|Ay\|_\infty > \|y\|_\infty$.

for all $b \geq b^2$, $\delta_M(b) \leq C_M b$. Recall also that for all $i \in \{0,...,k-1\}$,

$$\begin{aligned}
\hat{T}_i(y) &= E[\delta_W(\gamma_M a_M(M_{n-k+i+1:n}) + \hat{\mathscr{F}}(M_{n-k+i+1:n}) \cdot y) \\
&\quad - \delta_W(\gamma_M a_M(M_{n-k+i:n}) + \hat{\mathscr{F}}(M_{n-k+i:n}) \cdot y)] \\
S_i(x) &= E[\delta_M(\gamma_W a_W(W_{i+1:k}) + \mathscr{G}(W_{i+1:k}) \cdot x) - \delta_M(\gamma_W a_W(W_{i:k}) + \mathscr{G}(W_{i:k}) \cdot x)].
\end{aligned}$$

In the present case of $n > k$, $\hat{\mathscr{F}}_0 = \mathscr{F}_{n-k} \not\equiv 0$, so that $\hat{T}(y)$ depends on all entries of y. $\hat{T}_0(y)$ has no effect on $S(\hat{T}(y))$ because $\mathscr{G}_0 \equiv 0$. Given a vector $v = (v_0,...,v_{k-1}) \in \mathbb{R}^k$, we write v_{-0} for the vector $(v_1,...,v_{k-1})$.

For each $y \in \mathbb{R}^k_+$, $\gamma_M a_M(M_{n-k+1:n}) + \hat{\mathscr{F}}(M_{n-k+1:n}) \cdot y$ is the smallest one of the terms $\gamma_M a_M(M_{n-k+i:n}) + \hat{\mathscr{F}}(M_{n-k+i:n}) \cdot y$, $i \in \{1,...,k\}$, that occur in the definition of $\hat{T}_{-0}$. Moreover, the smallest entry of $\hat{\mathscr{F}}(M_{n-k+1:n}) = (F_{n-k+j:n-1}(M_{n-k+1:n}))_{j=0,...,k-1}$ is $F_{n-1:n-1}(M_{n-k+1:n})$, because $F_{n-1:n-1}$ first-order stochastically dominates the distributions of the lower order statistics. This implies $\gamma_M a_M(M_{n-k+1:n}) + \hat{\mathscr{F}}(M_{n-k+1:n}) \cdot y \geq ||y||_\infty F_{n-1:n-1}(M_{n-k+1:n})$. We define

$$m(b) := \begin{cases} F^{-1}_{n-1:n-1}\left(\frac{b^1}{b}\right) & \text{if } \ b \geq b^1 \\ \overline{m} & \text{if } \ 0 \leq b \leq b^1. \end{cases}$$

For $b \geq b^1$, $m(b)$ is strictly decreasing and satisfies $bF_{n-1:n-1}(m(b)) = b^1$. In particular, $\lim_{b\to\infty} m(b) = \underline{m}$. We write $\hat{T}_i(y) = A^{\hat{T}}_i(y) + \varepsilon^{\hat{T}}_i(y)$, where

$$\begin{aligned}
A^{\hat{T}}_i(y) &= E[(\delta_W(\gamma_M a_M(M_{n-k+i+1:n}) + \hat{\mathscr{F}}(M_{n-k+i+1:n}) \cdot y) \\
&\quad - \delta_W(\gamma_M a_M(M_{n-k+i:n}) + \hat{\mathscr{F}}(M_{n-k+i:n}) \cdot y))I_{\{M_{n-k+1:n} \geq m(||y||_\infty)\}}] \\
\varepsilon^{\hat{T}}_i(y) &= E[(\delta_W(\gamma_M a_M(M_{n-k+i+1:n}) + \hat{\mathscr{F}}(M_{n-k+i+1:n}) \cdot y) \\
&\quad - \delta_W(\gamma_M a_M(M_{n-k+i:n}) + \hat{\mathscr{F}}(M_{n-k+i:n}) \cdot y))I_{\{M_{n-k+1:n} < m(||y||_\infty)\}}].
\end{aligned}$$

Here, $I_{\{\cdot\}}$ is an indicator function. By the above definitions, $A^{\hat{T}}_i(y) = 0$ if $\|y\|_\infty \leq b^1$. Moreover, if $\|y\|_\infty \geq b^1$ and conditional on the event $\{M_{n-k+1:n} \geq m(||y||_\infty)\}$, we have $||y||_\infty F_{n-1:n-1}(M_{n-k+1:n}) \geq ||y||_\infty F_{n-1:n-1}(m(||y||_\infty)) = b^1$, so that the mean value theorem and $\delta_W'(b) \leq C_W$ for $b > b^1$ imply the following upper bound for $A_i^{\hat{T}}(y)$:

$$\begin{aligned}
&A_i^{\hat{T}}(y) \leq C_W E[(\gamma_M(a_M(M_{n-k+i+1:n}) - a_M(M_{n-k+i:n})) \\
&+ (\hat{\mathscr{F}}(M_{n-k+i+1:n}) - \hat{\mathscr{F}}(M_{n-k+i:n})) \cdot y) I_{\{M_{n-k+1:n} \geq m(\|y\|_\infty)\}}] \\
&\leq C_W \left(\gamma_M \hat{\Delta}_{M,i}^{a_M} + \sum_{j=0}^{k-1} E[F_{n-k+j:n-1}(M_{n-k+i+1:n}) - F_{n-k+j:n-1}(M_{n-k+i:n})] y_j \right) \\
&= C_W \left(\gamma_M \hat{\Delta}_{M,i}^{a_M} + (\hat{\Theta}_{n,k} y)_i \right).
\end{aligned}$$

Next, for all $i \in \{1, ..., k-1\}$, $\varepsilon_i^{\hat{T}}(y) = o(||y||_\infty)$ as $||y||_\infty \to \infty$. Indeed, $\varepsilon_i^{\hat{T}}(y) > 0$ and

$$\begin{aligned}
&\sum_{i=1}^{k-1} \varepsilon_i^{\hat{T}}(y) \leq E[(\delta_W(\gamma_M a_M(M_{n:n}) + \hat{\mathscr{F}}(M_{n:n}) \cdot y)) I_{\{M_{n-k+1:n} < m(||y||_\infty)\}}] \\
&\leq \delta_W(\gamma_M a_M(\overline{m}) + k||y||_\infty) E[I_{\{M_{n-k+1:n} < m(||y||_\infty)\}}].
\end{aligned}$$

As $||y||_\infty \to \infty$, the first factor is of order $O(||y||_\infty)$, and the second is of order $o(1)$. Thus, their product is of order $o(||y||_\infty)$. It follows that

$$\hat{T}_i(y) \leq C_W(\hat{\Theta}_{n,k} y)_i + o(||y||_\infty). \tag{24}$$

Consider S now. $\gamma_W a_W(W_{1:k}) + \mathscr{G}(W_{1:k}) \cdot x$ is the smallest non-zero term among the terms $\gamma_W a_W(W_{i:k}) + \mathscr{G}(W_{i:k}) \cdot x = \gamma_W a_W(W_{i:k}) + \mathscr{G}_{-0}(W_{i:k}) \cdot x_{-0}$, $i \in \{0, ..., k\}$, that occur in the definition of S (and $\gamma_W a_W(W_{0:k}) + \mathscr{G}(W_{0:k}) \cdot x = 0$). Moreover, $\gamma_W a_W(W_{1:k}) + \mathscr{G}_{-0}(W_{1:k}) \cdot x_{-0} \geq ||x_{-0}||_\infty G_{k-1:k-1}(W_{1:k})$. For $l \in \{1, 2\}$, we define:

$$w^l(b) := \begin{cases} G_{k-1:k-1}^{-1}\left(\frac{b^l}{b}\right) & \text{if } b \geq b^l \\ \overline{w} & \text{if } 0 \leq b \leq b^l, \end{cases}$$

i.e., for $b \geq b^l$, $bG_{k-1:k-1}(w^l(b)) = b^l$. For $i \geq 1$, we write $S_i(x) = A_i^S(x) + \varepsilon_i^S(x)$, where

$$\begin{aligned}
A_i^S(x) &= E[(\delta_M(\gamma_W a_W(W_{i+1:k}) + \mathscr{G}_{-0}(W_{i+1:k}) \cdot x_{-0}) \\
&\quad - \delta_M(\gamma_W a_W(W_{i:k}) + \mathscr{G}_{-0}(W_{i:k}) \cdot x_{-0})) I_{\{W_{1:k} \geq w^1(||x_{-0}||_\infty)\}}], \\
\varepsilon_i^S(x) &= E[(\delta_M(\gamma_W a_W(W_{i+1:k}) + \mathscr{G}_{-0}(W_{i+1:k}) \cdot x_{-0}) \\
&\quad - \delta_M(\gamma_W a_W(W_{i:k}) + \mathscr{G}_{-0}(W_{i:k}) \cdot x_{-0})) I_{\{W_{1:k} < w^1(||x_{-0}||_\infty)\}}]
\end{aligned}$$

Analogously to the estimates for $\hat{T}$, it follows that

$$A_i^S(x) \leq C_M \left(\gamma_W \Delta_{W,i}^{a_W} + (\Theta_k x)_i \right),$$

and $\varepsilon_i^S(x) = o(||x_{-0}||_\infty)$ as $||x_{-0}||_\infty \to \infty$. Next, we use the function w^2 to write $S_0(x) = A_0^S(x) + \varepsilon_0^S(x)$, where

$$\begin{aligned}
A_0^S(x) &= E[(\delta_M(\gamma_W a_W(W_{1:k}) + \mathscr{G}_{-0}(W_{1:k}) \cdot x_{-0}))I_{\{W_{1:k} \geq w^2(||x_{-0}||_\infty)\}}], \\
\varepsilon_0^S(x) &= E[(\delta_M(\gamma_W a_W(W_{1:k}) + \mathscr{G}_{-0}(W_{1:k}) \cdot x_{-0}))I_{\{W_{1:k} < w^2(||x_{-0}||_\infty)\}}].
\end{aligned}$$

Then $\varepsilon_0^S(x) = o(||x_{-0}||_\infty)$, and $A_0^S(x) \leq C_M \left(\gamma_W \Delta_{W,0}^{a_W} + (\Theta_k x)_i\right)$. Combining all estimates for S, we get for all $i \in \{0, ..., k-1\}$:

$$S_i(x) \leq C_M(\Theta_k x)_i + o(||x_{-0}||_\infty). \tag{25}$$

From (24) and (25), we get $S(\hat{T}(y)) \leq C_M C_W \Theta_k \hat{\Theta}_{n,k} y + R(y)$, where the remainder term R satisfies $R(y) = o(||y||_\infty)$ and thus (as all norms on $\mathbb{R}^k$ are equivalent) $R(y) = o(||y||)$ for any norm $\|\cdot\|$. This shows (22) and concludes the proof for the case $n > k$.

The proof for $n = k$ is analogous, and quite a bit simpler. In this case, both y_0 (corresponding to $\Delta_{W,0}^{\delta_M \circ b_W}$) and x_0 (corresponding to $\hat{\Delta}_{M,0}^{\delta_W \circ b_M}$) are irrelevant, so that one needs only the bounds $\delta_M'(b) \leq C_M$ and $\delta_W'(b) \leq C_W$ for $b \geq b^1$ to derive the analog of (22). Moreover, all row sums of Θ_n^2 are the same, so that the usual $|||\cdot|||_\infty$ norm can be used for the contraction argument.

Case $r(n,k)(\lim_{b\to\infty} \delta_M'(b))(\lim_{b\to\infty} \delta_W'(b)) \geq 1$: The proof of Corollary 2 shows that the fixed point vector would have to be coordinate-wise larger than the one for the case of linear benefits $d_M = \lim_{b\to\infty} \delta_M'(b)$ and $d_W = \lim_{b\to\infty} \delta_W'(b)$. However, by Theorem 1, no fixed point exists in this case. $\square$

Lemma 4. $\Delta_W^{a_W}$ *and* $\Delta_M^{a_M}$ *satisfy*

$$\begin{aligned}
\Delta_{W,i}^{a_W} &= \sum_{j=1}^{k-1} \theta_{k,ij} E[W_{i+j:2k-1}] \hat{\Delta}_{M,j}^{I_1} && \textit{for } i \in \{0, ..., k-1\}, \\
\Delta_{M,i}^{a_M} &= \sum_{j=0}^{k-1} \theta_{n,i(n-k+j)} E[M_{i+n-k+j:2n-1}] \Delta_{W,j}^{I_1} && \textit{for } i \in \{0, ..., n-1\}.
\end{aligned}$$

Proof of Lemma 4. As $wG_{j:k-1}(w) - \int_{\underline{w}}^{w} G_{j:k-1}(s)ds = \int_{\underline{w}}^{w} sg_{j:k-1}(s)ds$ (for $j \geq 1$), we find for $i \geq 1$:

$$
\begin{aligned}
\Delta_{W,i}^{a_W} &= E[a_W(W_{i+1:k}) - a_W(W_{i:k})] \\
&= \sum_{j=1}^{k-1} E\left[\int_{\underline{w}}^{W_{i+1:k}} s g_{j:k-1}(s)ds - \int_{\underline{w}}^{W_{i:k}} s g_{j:k-1}(s)ds\right] \hat{\Delta}_{M,j}^{I_1} \\
&= \sum_{j=1}^{k-1} \left(\int_{\underline{w}}^{\overline{w}} \left(\int_{\underline{w}}^{w} s g_{j:k-1}(s)ds\right) (g_{i+1:k}(w) - g_{i:k}(w))\, dw\right) \hat{\Delta}_{M,j}^{I_1} \\
&= \sum_{j=1}^{k-1} \left(\int_{\underline{w}}^{\overline{w}} w g_{j:k-1}(w)\, (G_{i:k}(w) - G_{i+1:k}(w))\, dw\right) \hat{\Delta}_{M,j}^{I_1} \\
&= \sum_{j=1}^{k-1} \left(\int_{\underline{w}}^{\overline{w}} w(k-1)\binom{k-2}{j-1}\binom{k}{i} G(w)^{i+j-1}(1-G(w))^{2k-1-(i+j)} g(w)dw\right) \hat{\Delta}_{M,j}^{I_1} \\
&= \sum_{j=1}^{k-1} \theta_{k,ij} \left(\int_{\underline{w}}^{\overline{w}} w(2k-1)\binom{2k-2}{i+j-1} G(w)^{i+j-1}(1-G(w))^{2k-1-(i+j)} g(w)dw\right) \hat{\Delta}_{M,j}^{I_1} \\
&= \sum_{j=1}^{k-1} \theta_{k,ij} E[W_{i+j:2k-1}] \hat{\Delta}_{M,j}^{I_1}.
\end{aligned}
$$

Similarly, for $\Delta_{M,i}^{a_M}$, $i \geq 1$ (if $n = k$, the term for $j = 0$ in the following calculation is trivially equal to zero):

$$
\begin{aligned}
\Delta_{M,i}^{a_M} &= E[a_M(M_{i+1:n}) - a_M(M_{i:n})] \\
&= \sum_{j=0}^{k-1} E\left[\int_{\underline{m}}^{M_{i+1:n}} s f_{n-k+j:n-1}(s)ds - \int_{\underline{m}}^{M_{i:n}} s f_{n-k+j:n-1}(s)ds\right] \Delta_{W,j}^{I_1} \\
&= \sum_{j=0}^{k-1} \left(\int_{\underline{m}}^{\overline{m}} \left(\int_{\underline{m}}^{m} s f_{n-k+j:n-1}(s)ds\right) (f_{i+1:n}(m) - f_{i:n}(m))\, dm\right) \Delta_{W,j}^{I_1} \\
&= \sum_{j=0}^{k-1} \left(\int_{\underline{m}}^{\overline{m}} m f_{n-k+j:n-1}(m)\, (F_{i:n}(m) - F_{i+1:n}(m))\, dm\right) \Delta_{W,j}^{I_1} \\
&= \sum_{j=0}^{k-1} \theta_{n,i(n-k+j)} E[M_{i+n-k+j:2n-1}] \Delta_{W,j}^{I_1}.
\end{aligned}
$$

The proofs for $i = 0$ follow from analogous calculations (using $a_W(W_{0:k}) = a_M(M_{0:n}) = 0$ and $1 - G_{1:k}(\overline{w}) = 1 - F_{1:n}(\overline{m}) = 0$). □

Lemma 5. *If F has increasing virtual valuations, i.e., $m - \frac{1-F(m)}{f(m)}$ is weakly increasing, and if $\underline{m} = 0$, then $4E[M_{1:3}] \geq E[M_{2:3}]$. The inequality is satisfied with equality if virtual valuations are constant.*

Proof of Lemma 5. Recall $F_{1:3} = 1-(1-F)^3$ and $F_{1:3} - F_{2:3} = 3F(1-F)^2$. We have

$$\begin{aligned}
4E[M_{1:3}] - E[M_{2:3}] &= 3E[M_{1:3}] + (E[M_{1:3}] - E[M_{2:3}]) \\
&= 3\int_0^{\overline{m}} (1-F(x))^3 dx - 3\int_0^{\overline{m}} F(x)(1-F(x))^2 dx \\
&= 3\int_0^{\overline{m}} (1-F(x))^2(1-2F(x))dx \\
&= 3\left(\int_0^{\overline{m}} (1-F(x))^2 dx - \int_0^{\overline{m}} (1-F(x))^2 2F(x)dx\right).
\end{aligned}$$

Defining $h = (1-F)^2/f$ and letting M denote a random variable with c.d.f. F, we get

$$\begin{aligned}
4E[M_{1:3}] - E[M_{2:3}] &= 3\int_0^{\overline{m}} h(x)f(x)dx - \int_0^{\overline{m}} h(x)2F(x)f(x)dx \\
&= 3E[h(M)] - 3E[h(M_{2:2})]
\end{aligned}$$

where we used that $f_{2:2} = 2Ff$. Thus, if h is decreasing, $4E[M_{1:3}] - E[M_{2:3}]$ is non-negative. Yet this is indeed the case since $1/h$ is the so-called zoom rate associated with F and since it is shown in Ewerhart (2013) that the signs of the derivatives of the zoom rate and of the virtual valuation function coincide, see also Szech (2011). $\square$

Proof of Corollary 1. Consider the equilibrium strategies $\widetilde{b}_M$ and $\widetilde{b}_W$ for the case of linear benefit functions $\widetilde{\delta}_M(\beta_W) := \delta_M'(0)\beta_W$ and $\widetilde{\delta}_W(\beta_M) := \delta_W'(0)\beta_M$ (the equilibrium exists, by Theorem 1, because $r(n,k)\delta_M'(0)\delta_W'(0) < 1$). The limit conditions on type supports imply $\widetilde{b}_M(\overline{m}) \to 0$ and $\widetilde{b}_W(\overline{w}) \to 0$. Indeed, if $n = k$, then $a_M(\overline{m}) \leq \overline{m}(\overline{w} - \underline{w})$ and $a_W(\overline{w}) \leq (\overline{m} - \underline{m})\overline{w}$, and if $n > k$, then $a_M(\overline{m}), a_W(\overline{w}) \leq \overline{mw}$. Thus, $a_M(\overline{m}) \to 0$ and $a_W(\overline{w}) \to 0$, so that $\Delta_M^{a_M}, \Delta_W^{a_W} \to 0$. Thus, using (12), it follows that $\Delta_W^{\widetilde{b}_W} = (I_k - \delta_M'(0)\delta_W'(0)\Theta_k\hat{\Theta}_{n,k})^{-1}(\gamma_W\Delta_W^{a_W} + d_W\gamma_M\Theta_k\hat{\Delta}_M^{a_M}) \to 0$ (Θ_k and $\hat{\Theta}_{n,k}$ are fix). Similarly, $\Delta_M^{\widetilde{b}_M} \to 0$ (using (11)). Hence, $\widetilde{b}_M(\overline{m}) = \gamma_M a_M(\overline{m}) + \delta_M'(0)\hat{\mathscr{F}}(\overline{m}) \cdot \Delta_W^{\widetilde{b}_W} \to 0$, and $\widetilde{b}_W(\overline{w}) = \gamma_W a_W(\overline{w}) + \delta_W'(0)\mathscr{G}(\overline{w}) \cdot \hat{\Delta}_M^{\widetilde{b}_M} \to 0$.

To complete the proof, we show that for any equilibrium of the matching contest with benefit functions δ_M and δ_W, $b_M(m) \leq \widetilde{b}_M(m)$ for all $m \in [\underline{m}, \overline{m}]$ and $b_W(w) \leq \widetilde{b}_W(w)$ for all $w \in [\underline{w}, \overline{w}]$. First, the concavity of δ_W and δ_M, the mean value theorem, and the identities (6) and (7) imply the following bounds for $\hat{\Delta}_M^{\delta_W \circ b_M}$ and $\Delta_W^{\delta_M \circ b_W}$:

$$\begin{aligned}
\hat{\Delta}_M^{\delta_W \circ b_M} &\leq \delta_W'(0)(\gamma_M\hat{\Delta}_M^{a_M} + \hat{\Theta}_{n,k}\Delta_W^{\delta_M \circ b_W}), \\
\Delta_W^{\delta_M \circ b_W} &\leq \delta_M'(0)(\gamma_W\Delta_W^{a_W} + \Theta_k\hat{\Delta}_M^{\delta_W \circ b_M}).
\end{aligned}$$

Thus (as all entries of Θ_k are non-negative),

$$(I_k - \delta_M'(0)\delta_W'(0)\Theta_k\hat{\Theta}_{n,k})\Delta_W^{\delta_M \circ b_W} \leq \delta_M'(0)(\gamma_W \Delta_W^{a_W} + \delta_W'(0)\gamma_M \Theta_k \hat{\Delta}_M^{a_M}). \tag{26}$$

Applying $(I_k - \delta_M'(0)\delta_W'(0)\Theta_k\hat{\Theta}_{n,k})^{-1} = \sum_{l=0}^{\infty}(\delta_M'(0)\delta_W'(0)\Theta_k\hat{\Theta}_{n,k})^l \geq 0$ to both sides of the vector inequality (26) yields

$$\Delta_W^{\delta_M \circ b_W} \leq \delta_M'(0)\Delta_W^{\widetilde{b}_W} = \Delta_W^{\widetilde{\delta}_M \circ \widetilde{b}_W}.$$

$\hat{\Delta}_M^{\delta_W \circ b_M} \leq \hat{\Delta}_M^{\widetilde{\delta}_W \circ \widetilde{b}_M}$ follows from an entirely analogous argument. $b_M(m) \leq \widetilde{b}_M(m)$ now follows from (4), and $b_W(w) \leq \widetilde{b}_W(w)$ follows from (5). $\square$

Proof of Corollary 2. Consider the expression (6) for $\Delta_{M,i}^{\delta_W \circ b_M}$ ($i \in \{0,...,n-1\}$). Invoking the concavity of δ_W and the mean value theorem, we obtain the following lower bound:

$$\Delta_{M,i}^{\delta_W \circ b_M} \geq \delta_W'(b_M(\overline{m}))(\gamma_M \Delta_{M,i}^{a_M} + (\Theta_{n,k}\Delta_W^{\delta_M \circ b_W})_i).$$

Analogously, (7), the concavity of δ_M, and the mean value theorem imply

$$\Delta_{W,i}^{\delta_M \circ b_W} \geq \delta_M'(b_W(\overline{w}))(\gamma_W \Delta_{W,i}^{a_W} + (\Theta_k \hat{\Delta}_M^{\delta_W \circ b_M})_i),$$

for all $i \in \{0,...,k-1\}$. Thus,

$$\begin{aligned}
\hat{\Delta}_M^{\delta_W \circ b_M} &\geq \delta_W'(b_M(\overline{m}))(\gamma_M \hat{\Delta}_M^{a_M} + \hat{\Theta}_{n,k}\Delta_W^{\delta_M \circ b_W}), \\
\Delta_W^{\delta_M \circ b_W} &\geq \delta_M'(b_W(\overline{w}))(\gamma_W \Delta_W^{a_W} + \Theta_k \hat{\Delta}_M^{\delta_W \circ b_M}).
\end{aligned}$$

As all entries of Θ_k are non-negative, we get:

$$\begin{aligned}
&(I_k - \delta_M'(b_W(\overline{w}))\delta_W'(b_M(\overline{m}))\Theta_k\hat{\Theta}_{n,k})\Delta_W^{\delta_M \circ b_W} \\
&\quad \geq \quad \delta_M'(b_W(\overline{w}))(\gamma_W \Delta_W^{a_W} + \delta_W'(b_M(\overline{m}))\gamma_M \Theta_k \hat{\Delta}_M^{a_M}).
\end{aligned} \tag{27}$$

If $\delta_M'(b_W(\overline{w})) = 0$, then the claim of the theorem holds trivially. Otherwise, the right hand side of (27) is a positive vector. Thus, $\Delta_W^{\delta_M \circ b_W}$ solves

$$(I_k - \delta_M'(b_W(\overline{w}))\delta_W'(b_M(\overline{m}))\Theta_k\hat{\Theta}_{n,k})\Delta_W^{\delta_M \circ b_W} = z,$$

for some $z > 0$. In the proof of Theorem 1, we have shown that this is possible if and only if $r(n,k)\delta_M'(b_W(\overline{w}))\delta_W'(b_M(\overline{m})) < 1$. $\square$

Proof of Lemma 3. We give the proof for U_M and $n = k$. The remaining cases are analogous. Invoking (4) and the formula for a_M, we find:

$$
\begin{aligned}
U_M(m) &= \gamma_M m\Psi(m) + E[\delta_M(b_W(W_{1:n}))] + \hat{\mathscr{F}}(m)\cdot\Delta_W^{\delta_M\circ b_W} - b_M(m)\\
&= \gamma_M m\left(E[W_{1:n}] + \hat{\mathscr{F}}(m)\cdot\Delta_W^{I_1}\right) + E[\delta_M(b_W(W_{1:n}))] - \gamma_M a_M(m)\\
&= E[\delta_M(b_W(W_{1:n}))] + \gamma_M\left(mE[W_{1:n}] + \int_{\underline{m}}^{m}\hat{\mathscr{F}}(s)ds\cdot\Delta_W^{I_1}\right)\\
&= E[\delta_M(b_W(W_{1:n}))] + \gamma_M\underline{m}E[W_{1:n}] + \gamma_M\int_{\underline{m}}^{m}\Psi(s)ds.
\end{aligned}
$$

□

Proof of Theorem 3. The maximization problems for types m and w, who assume that others invest according to continuous, non-decreasing functions b_M and b_W that are strictly increasing and differentiable on $[m_r,\overline{m}]$ and $[\underline{w},\overline{w}]$ and satisfy $b_M(m_r) = b_W(0) = 0$, are given by:

$$
\begin{aligned}
&\max_{s\in[0,\overline{m}]}\left[\gamma_M m\psi_r(s) - b_M(s) + \delta_M(b_W(\psi_r(s)))\right], \text{ and}\\
&\max_{s\in[0,\overline{w}]}\left[\gamma_W w\phi_r(s) - b_W(s) + \delta_W(b_M(\phi_r(s)))\right].
\end{aligned}
$$

This implies the following necessary conditions for equilibrium investments by types $m > m_r$ and $w > 0$:

$$
\begin{aligned}
b_M'(m) &= \gamma_M m\psi_r'(m) + \delta_M'(b_W(\psi_r(m)))b_W'(\psi_r(m))\psi_r'(m)\\
b_W'(w) &= \gamma_W w\phi_r'(w) + \delta_W'(b_M(\phi_r(w)))b_M'(\phi_r(w))\phi_r'(w).
\end{aligned}
$$

Evaluating the second condition at $w = \psi_r(m)$, multiplying it by $\psi_r'(m) > 0$, and using $\phi_r(\psi_r(m)) = m$ and $\phi_r'(\psi_r(m))\psi_r'(m) = 1$, we obtain the following system of first-order ODE for b_m and $b_W\circ\psi_r$:

$$
\begin{aligned}
b_M'(m) &= \gamma_M m\psi_r'(m) + \delta_M'((b_W\circ\psi_r)(m)))(b_W\circ\psi_r)'(m)\\
(b_W\circ\psi_r)'(m) &= \gamma_W\psi_r(m) + \delta_W'(b_M(m))b_M'(m).
\end{aligned}
\tag{28}
$$

If $\delta_M'(0)\delta_W'(0) < 1$, we may rewrite the system (28) equivalently in standard form

$$
\begin{aligned}
b_M'(m) &= \frac{\gamma_M m\psi_r'(m) + \delta_M'((b_W\circ\psi_r)(m))\gamma_W\psi_r(m)}{1-\delta_W'(b_M(m))\delta_M'((b_W\circ\psi_r)(m))}\\
(b_W\circ\psi_r)'(m) &= \frac{\gamma_W\psi_r(m) + \delta_W'(b_M(m))\gamma_M m\psi_r'(m)}{1-\delta_W'(b_M(m))\delta_M'((b_W\circ\psi_r)(m))}.
\end{aligned}
$$

$\delta_M'(0)\delta_W'(0) < 1$ and the continuity of δ_M'' and δ_W'' imply that the Lipschitz condition guaranteeing a unique pair of continuously differentiable solutions $(b_M, b_W \circ \psi_r)$ with initial values $b_M(m_r) = b_W(\psi_r(m_r)) = 0$ is satisfied, and clearly both functions are strictly increasing. Sufficiency conditions for optimality follow immediately from supermodularity. The formulas for $u_M^{(r)}$ and $u_W^{(r)}$ follow from $u_M^{(r)}(m_r) = u_W^{(r)}(0) = 0$ and payoff equivalence. This proves part (i).

For part (ii), note that if b_M and b_W are strictly increasing functions, they are differentiable almost everywhere. In particular, there is a set of full measure in $[m_r, \overline{m}]$ where both b_M and $b_W \circ \psi_r$ are differentiable. But, if b_M and b_W are equilibrium strategies, the system (28) must be satisfied for each of these types. However, for $\delta_M' \equiv 1$ and $\delta_W' \equiv 1$, (28) is violated for any m. This concludes the proof. $\square$

The proofs of Theorems 4 and 5 use the following well-known result about the asymptotic distribution of central order statistics:

Lemma 6. *For $m \in (\underline{m}, \bar{m})$, set $p = F(m) > 0$. Let $\to^d$ denote convergence in distribution, as $n \to \infty$. If $(j_n)_{n\in\mathbb{N}}$ is a sequence that satisfies $\frac{j_n}{n} - p = o(n^{-1/2})$, then*

$$\sqrt{n}(M_{j_n:n} - m) \to^d N\left(0, \frac{p(1-p)}{f(m)^2}\right),$$

In particular, the result applies for $j_n = \lceil pn \rceil$, the ceiling *of pn (the smallest integer greater than or equal to pn).*

Proof of Lemma 6. See Theorem 10.3 in David and Nagaraja (2003). $\square$

Lemma 7. *i) For all $m \in [0, \overline{m}]$ and $w \in [0, \overline{w}]$:*

$$\lim_{k\to\infty, \frac{k}{n_k}\to 1-r} \Psi^{(n_k,k)}(m) = \psi_r(m) \ \ \textit{and} \ \ \lim_{k\to\infty, \frac{k}{n_k}\to 1-r} \Phi^{(n_k,k)}(w) = \phi_r(w).$$

ii) For all $m \in [0, \overline{m}]$ and $w \in [0, \overline{w}]$:

$$\lim_{k\to\infty, \frac{k}{n_k}\to 1-r} \int_0^m \Psi^{(n_k,k)}(s)ds = \int_0^m \psi_r(s)ds \ \ \textit{and} \ \ \lim_{k\to\infty, \frac{k}{n_k}\to 1-r} \int_0^w \Phi^{(n_k,k)}(s)ds = \int_0^w \phi_r(s)ds.$$

Proof of Lemma 7. i) We give the proof for $\lim_{k\to\infty, \frac{k}{n_k}\to 1} \Psi^{(n_k,k)}(m) = \psi_0(m)$, i.e., for the convergence of a given type of man's expected partner when $r = 0$. The proof for $r \in (0,1)$ is a bit more cumbersome in terms of notation, but otherwise analogous. The proof for the convergence of $\Phi^{(n_k,k)}$ is analogous as well. Consider any $m \in (0, \bar{m})$ and set $p = F(m) > 0$. Fixing an arbitrary $\varepsilon > 0$ (with $m - \varepsilon \in (0, \bar{m})$ and

$m+\varepsilon \in (0,\bar{m})$), the convergence in law of $\sqrt{n_k}(M_{\lceil F(m-\varepsilon)n_k\rceil:n_k-1} - (m-\varepsilon))$ (by Lemma 6) implies that $F_{\lceil F(m-\varepsilon)n_k\rceil:n_k-1}(m)$ converges exponentially fast to 1 (in k, or equivalently n_k). Moreover $F_{j:n_k-1}(m) \geq F_{\lceil F(m-\varepsilon)n_k\rceil:n_k-1}(m)$ for all $j \leq \lceil F(m-\varepsilon)n\rceil$ (by stochastic dominance). Similarly, $F_{\lceil F(m+\varepsilon)n_k\rceil:n_k-1}(m)$ converges exponentially fast to 0, and $F_{j:n_k-1}(m) \leq F_{\lceil F(m+\varepsilon)n_k\rceil:n_k-1}(m)$ for all $j \geq \lceil F(m+\varepsilon)n_k\rceil$. It follows that

$$\Psi^{(n_k,k)}(m) = \begin{cases} E[W_{1:n_k}] + \sum_{j=1}^{n_k-1} F_{j:n_k-1}(m)E[W_{j+1:n_k} - W_{j:n_k}] & \text{if } n_k = k \\ \sum_{j=0}^{k-1} F_{n_k-k+j:n_k-1}(m)E[W_{j+1:k} - W_{j:k}] & \text{if } n_k > k \end{cases}$$

can be written as the sum of a number in $\left[E[W_{\lceil F(m-\varepsilon)n_k\rceil+k-n_k:k}], E[W_{\lceil F(m+\varepsilon)n_k\rceil+k-n_k:k}]\right]$ and an error term that converges exponentially fast to zero. Moreover, as a simple consequence of Lemma 6 (applied to women's order statistics), we have:

$$\lim_{k\to\infty,\frac{k}{n_k}\to 1} E[W_{\lceil F(m-\varepsilon)n_k\rceil+k-n_k:k}] = G^{-1}(F(m-\varepsilon)),$$
$$\lim_{k\to\infty,\frac{k}{n_k}\to 1} E[W_{\lceil F(m+\varepsilon)n_k\rceil+k-n_k:k}] = G^{-1}(F(m+\varepsilon)).$$

Thus, it follows that

$$G^{-1}(F(m-\varepsilon)) \leq \liminf_{k\to\infty,\frac{k}{n_k}\to 1} \Psi^{(n_k,k)}(m) \leq \limsup_{k\to\infty,\frac{k}{n_k}\to 1} \Psi^{(n_k,k)}(m) \leq G^{-1}(F(m+\varepsilon)).$$

Letting $\varepsilon \to 0$ then yields $\lim_{k\to\infty,\frac{k}{n_k}\to 1} \Psi^{(n_k,k)}(m) = \psi_0(m)$, for all $m \in (0,\bar{m})$. The result for $m=0$ and $m=\bar{m}$ now follows immediately from the monotonicity and boundedness of $\Psi^{(n_k,k)}$.

ii) This follows from (i) and the Dominated Convergence Theorem. □

We will use the following two lemmas in the proofs of Theorems 4 and 5. Lemma 8 is due to Varah (1975).

Lemma 8 (Varah, 1975)**.** *Let A be a $l\times l$-matrix that is diagonally dominant by rows. That is, for all $i=0,...,l-1$, $|a_{i,i}| > \sum_{j\neq i}|a_{i,j}|$. Set $\alpha := \min_i(|a_{i,i}| - \sum_{j\neq i}|a_{i,j}|)$. Then $||A^{-1}||_\infty \leq \frac{1}{\alpha}$.*

Lemma 9. *$||\Delta_W^{a_W}||_\infty = O\left(\frac{1}{n}\right)$, and if Condition 1 is satisfied also $||\Delta_M^{a_M}||_\infty = O\left(\frac{1}{k}\right)$.*

Proof of Lemma 9. Let $c_f := \min_{m\in[\underline{m},\overline{m}]} f(m) > 0$ and $c_g := \min_{w\in[\underline{w},\overline{w}]} g(w) > 0$. We will prove the lemma using the representations of $\Delta_W^{a_W}$ and $\Delta_M^{a_M}$ from Lemma 4. Note first that, for all $i \in \{0, ..., n-1\}$,

$$
\begin{aligned}
\Delta_{M,i}^{I_1} = E[M_{i+1:n} - M_{i:n}] &= \int_{\underline{m}}^{\overline{m}} (F_{i:n}(m) - F_{i+1:n}(m))dm \;+\; \mathbb{I}_{\{i=0\}}\underline{m} \\
&= \int_{\underline{m}}^{\overline{m}} \binom{n}{i} F^i(m)(1-F(m))^{n-i} dm \;+\; \mathbb{I}_{\{i=0\}}\underline{m} \\
&\leq \frac{1}{c_f} \int_{\underline{m}}^{\overline{m}} \binom{n}{i} F^i(m)(1-F(m))^{n-i} f(m) dm \;+\; \mathbb{I}_{\{i=0\}}\underline{m} \\
&= \frac{1}{c_f(n+1)} \;+\; \mathbb{I}_{\{i=0\}}\underline{m},
\end{aligned}
$$

where $\mathbb{I}_{\{\cdot\}}$ is the usual indicator function. An analogous inequality applies, of course, for $\Delta_{W,i}^{I_1}$, $i \in \{0, ..., k-1\}$. Using Lemma 4 and Lemma 2 (ii), we obtain:

$$
\begin{aligned}
\Delta_{W,i}^{a_W} &= \sum_{j=1}^{k-1} \theta_{k,ij} E[W_{i+j:2k-1}] \hat{\Delta}_{M,j}^{I_1} < \frac{\overline{w}(k-1)}{c_f(k+1)(n+1)} = O\left(\frac{1}{n}\right) \\
\Delta_{M,i}^{a_M} &= \sum_{j=0}^{k-1} \theta_{n,i(n-k+j)} E[M_{i+n-k+j:2n-1}] \Delta_{W,j}^{I_1} < \frac{\overline{m}(n-1)}{c_g(k+1)(n+1)} = O\left(\frac{1}{k}\right),
\end{aligned}
$$

where the second inequality uses Condition 1. $\square$

Proof of Theorem 4. Given the results of Lemma 3 and Lemma 7 ii), we still have to show

$$
\lim_{n\to\infty} E[\delta_M(b_W^{(n,n)}(W_{1:n}))] = \lim_{n\to\infty} E[\delta_W(b_M^{(n,n)}(M_{1:n}))] = 0 \quad \text{(balanced case)}
$$

and

$$
\lim_{k\to\infty, k<n_k, \frac{k}{n_k}\to 1-r} E[\delta_W(b_M^{(n_k,k)}(M_{n_k-k+1:n_k}))] = 0 \quad \text{(unbalanced case)}.
$$

As equilibrium strategies for arbitrary benefit functions satisfying $\delta_M'(0)\delta_W'(0) < 1$ are pointwise dominated by the equilibrium strategies for the case of linear benefit functions with $d_M = \delta_M'(0)$ and $d_W = \delta_W'(0)$ (see the proof of Theorem 1), we need to consider only the latter case. Starting from (11), we obtain:

$$
\begin{aligned}
&||\hat{\Delta}_M^{b_M^{(n,k)}}||_\infty = ||(I_k - d_M d_W \hat{\Theta}_{n,k}\Theta_k)^{-1}(\gamma_M \hat{\Delta}_M^{a_M^{(n,k)}} + d_M \gamma_W \hat{\Theta}_{n,k} \Delta_W^{a_W^{(n,k)}})||_\infty \\
&\leq ||(I_k - d_M d_W \hat{\Theta}_{n,k}\Theta_k)^{-1}||_\infty (\gamma_M ||\hat{\Delta}_M^{a_M^{(n,k)}}||_\infty + d_M \gamma_W ||\hat{\Theta}_{n,k}||_\infty ||\Delta_W^{a_W^{(n,k)}}||_\infty) = O\left(\frac{1}{k}\right),
\end{aligned}
$$

where the last step uses $||(I_k - d_M d_W \hat{\Theta}_{n,k}\Theta_k)^{-1}||_\infty \leq \frac{1}{1-d_M d_W}$ (by Lemma 8, because $\hat{\Theta}_{n,k}\Theta_k$ is sub-stochastic), $||\hat{\Theta}_{n,k}||_\infty \leq 1$, as well as $||\hat{\Delta}_M^{a_M^{(n,k)}}||_\infty = O\left(\frac{1}{k}\right)$ and $||\Delta_W^{a_W^{(n,k)}}||_\infty =$

$O\left(\frac{1}{n}\right)$ (by Lemma 9). An analogous argument, starting from (12) shows $||\Delta_W^{b_W^{(n,k)}}||_\infty = O\left(\frac{1}{k}\right)$. As $E[b_W^{(n,n)}(W_{1:n})] = \Delta_{W,0}^{b_W^{(n,n)}}$ and $E[b_M^{(n,n)}(M_{1:n})] = \Delta_{M,0}^{b_M^{(n,n)}}$, this proves the claim of the Theorem for balanced markets.

We still have to show $\lim_{k\to\infty, k<n_k, \frac{k}{n_k}\to 1-r} E[b_M^{(n_k,k)}(M_{n_k-k+1:n_k}))] = 0$. Note that

$$\begin{aligned} b_M^{(n_k,k)}(m) &= \gamma_M a_M^{(n_k,k)}(m) + d_M \hat{\mathscr{F}}(m)\cdot \Delta_W^{b_W^{(n_k,k)}} \\ &\leq \hat{\mathscr{F}}(m)\cdot(\gamma_M m \Delta_W^{I_1} + d_M \Delta_W^{b_W^{(n_k,k)}}) = \sum_{j=0}^{k-1} F_{n_k-k+j:n_k-1}(m)(\gamma_M m \Delta_{W,j}^{I_1} + d_M \Delta_{W,j}^{b_W^{(n_k,k)}}). \end{aligned}$$

As $||\Delta_W^{I_1}||_\infty = O\left(\frac{1}{k}\right)$ (see the proof of Lemma 9) and $||\Delta_W^{b_W^{(n,k)}}||_\infty = O\left(\frac{1}{k}\right)$ (see above), the claim follows immediately from Lemma 6, which implies that for an arbitrary $\varepsilon > 0$ and any sequence $j_k \geq \varepsilon k$, the probability that $F_{n_k-k+j_k:n_k-1}(M_{n_k-k+1:n_k})$ is larger than ε converges exponentially fast to 0. $\square$

The proof of Theorem 5 i) uses the following lemma.

Lemma 10. *i) The diagonal entries of Θ_n satisfy:*

$$\begin{aligned} \operatorname{argmin}_{i\in\{1,\dots,n-1\}} \theta_{n,ii} &= \frac{n}{2} \text{ if } n \text{ is even}, \\ \operatorname{argmin}_{i\in\{1,\dots,n-1\}} \theta_{n,ii} &= \left\{\frac{n-1}{2}, \frac{n+1}{2}\right\} \text{ if } n \text{ is odd}. \end{aligned}$$

ii) For even n, $\lim_{n\to\infty} \theta_{n,\frac{n}{2}\frac{n}{2}}\sqrt{\pi n} = 1$*, and for odd n,* $\lim_{n\to\infty} \theta_{n,\frac{n-1}{2}\frac{n-1}{2}}\sqrt{\pi n} = 1$.

Proof of Lemma 10. i) According to Lemma 2 (i), for $i \in \{1,\dots,n-1\}$:

$$\theta_{n,ii} = \frac{n-1}{2n-1}\frac{\binom{n}{i}\binom{n-2}{i-1}}{\binom{2n-2}{2i-1}} = \frac{n-1}{2n-1}\frac{\binom{2i-1}{i}\binom{2n-1-2i}{n-i}}{\binom{2n-2}{n}}.$$

For $i \in \{1,\dots,n-2\}$, consider $\ln\theta_{n,ii} - \ln\theta_{n,(i+1)(i+1)}$.

$$\begin{aligned} \ln\theta_{n,ii} - \ln\theta_{n,(i+1)(i+1)} &= \left(\sum_{l=i+1}^{2i-1}\ln l - \sum_{l=1}^{i-1}\ln l + \sum_{l=n+1-i}^{2n-1-2i}\ln l - \sum_{l=1}^{n-1-i}\ln l\right) \\ &- \left(\sum_{l=i+2}^{2i+1}\ln l - \sum_{l=1}^{i}\ln l + \sum_{l=n-i}^{2n-3-2i}\ln l - \sum_{l=1}^{n-2-i}\ln l\right) = \ln(i+1) - \ln(2i) \\ &- \ln(2i+1) + \ln i - \ln(n-i) + \ln(2n-2-2i) + \ln(2n-1-2i) - \ln(n-1-i) \\ &= \ln(i+1) - \ln(2i+1) - \ln(n-i) + \ln(2n-1-2i) \\ &= \ln\left(2 - \frac{1}{n-i}\right) - \ln\left(2 - \frac{1}{i+1}\right). \end{aligned}$$

Thus, if n is even, we have $\theta_{n,ii} > \theta_{n,(i+1)(i+1)}$ if and only if $i \leq \frac{n}{2}-1$ (otherwise the strict reverse inequality holds). The claim for odd n also follows.

ii) This follows from Stirling's approximation, $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+O\left(\frac{1}{n}\right)\right)$. We spell out the case of even n.

$$
\begin{aligned}
\sqrt{\pi n}\,\theta_{n,\frac{n}{2}\frac{n}{2}} &= \sqrt{\pi n}\frac{n-1}{2n-1}\frac{n!(n-2)!(n-1)!^2}{\left(\frac{n}{2}\right)!^2\left(\frac{n-2}{2}\right)!^2(2n-2)!}\\
&= \sqrt{n}\frac{n-1}{2n-1}\frac{1}{\sqrt{2}}\frac{\sqrt{n(n-2)}(n-1)}{\frac{n}{2}\frac{n-2}{2}\sqrt{2n-2}}\frac{n^n(n-2)^{n-2}(n-1)^{2n-2}}{\left(\frac{n}{2}\right)^n\left(\frac{n-2}{2}\right)^{n-2}(2n-2)^{2n-2}}\frac{\left(1+O\left(\frac{1}{n}\right)\right)}{\left(1+O\left(\frac{1}{n}\right)\right)}\\
&= \sqrt{n}\frac{n-1}{2n-1}\frac{2\sqrt{n-1}}{\sqrt{n(n-2)}}\frac{\left(1+O\left(\frac{1}{n}\right)\right)}{\left(1+O\left(\frac{1}{n}\right)\right)}.
\end{aligned}
$$

This ratio converges to 1 as $n \to \infty$. □

Proof of Theorem 5. Part (i): We show $\lim_{n\to\infty}(E[b_M^{(n,n)}(M_{1:n})] - E[b_W^{(n,n)}(W_{1:n})]) = 0$, i.e., $\lim_{n\to\infty}\left(\Delta_{M,0}^{b_M^{(n,n)}} - \Delta_{W,0}^{b_W^{(n,n)}}\right) = 0$.

Given a vector $v = (v_0,...,v_{n-1}) \in \mathbb{R}^n$, we write v_{-0} for the vector $(v_1,...,v_{n-1})$. We define $V_n \in \mathbb{R}^n$ via $V_{n,j} := \theta_{n,0j}$ for $j \in \{0,...,n-1\}$, i.e., V_n is the first row vector of Θ_n. Recall also that $\hat{\Theta}_{n,n-1}$ is the $(n-1)\times(n-1)$ matrix that results from deleting the first column (which is zero) and the first row of Θ_n. (9) and (10) imply:

$$
\Delta_{M,0}^{b_M^{(n,n)}} = \gamma_M \Delta_{M,0}^{a_M^{(n,n)}} + V_{n,-0}\cdot\Delta_{W,-0}^{b_W^{(n,n)}} \quad\text{and}\quad \Delta_{W,0}^{b_W^{(n,n)}} = \gamma_W \Delta_{W,0}^{a_W^{(n,n)}} + V_{n,-0}\cdot\Delta_{M,-0}^{b_M^{(n,n)}}.
$$

Thus, $\Delta_{M,0}^{b_M^{(n,n)}} - \Delta_{W,0}^{b_W^{(n,n)}} = \gamma_M \Delta_{M,0}^{a_M^{(n,n)}} - \gamma_W \Delta_{W,0}^{a_W^{(n,n)}} + V_{n,-0}\cdot\left(\Delta_{W,-0}^{b_W^{(n,n)}} - \Delta_{M,-0}^{b_M^{(n,n)}}\right)$. Using that $||\Delta_M^{a_M^{(n,n)}}||_\infty = O\left(\frac{1}{n}\right)$ and $||\Delta_W^{a_W^{(n,n)}}||_\infty = O\left(\frac{1}{n}\right)$ (by Lemma 9), as well as $||V_{n,-0}||_1 = \frac{n-1}{n+1}$ (by Lemma 2 ii), $\lim_{n\to\infty}\left(\Delta_{M,0}^{b_M^{(n,n)}} - \Delta_{W,0}^{b_W^{(n,n)}}\right) = 0$ follows from

$$
\lim_{n\to\infty}||\Delta_{W,-0}^{b_W^{(n,n)}} - \Delta_{M,-0}^{b_M^{(n,n)}}||_\infty = 0,
$$

which we now show. First, using (again) that the first column of Θ_n is zero, (9) and (10) yield

$$
\begin{aligned}
\Delta_{M,-0}^{b_M^{(n,n)}} &= \gamma_M \Delta_{M,-0}^{a_M^{(n,n)}} + \hat{\Theta}_{n,n-1}\Delta_{W,-0}^{b_W^{(n,n)}},\\
\Delta_{W,-0}^{b_W^{(n,n)}} &= \gamma_W \Delta_{W,-0}^{a_W^{(n,n)}} + \hat{\Theta}_{n,n-1}\Delta_{M,-0}^{b_M^{(n,n)}}.
\end{aligned}
$$

Thus, analogous to (11) and (12), we find the following explicit representations of

$\Delta_{M,-0}^{b_M^{(n,n)}}$ and $\Delta_{W,-0}^{b_W^{(n,n)}}$ (using only the entries of $\hat{\Theta}_{n,n-1}$):

$$\Delta_{M,-0}^{b_M^{(n,n)}} = (I_{n-1} - \hat{\Theta}_{n,n-1}^2)^{-1}(\gamma_M \Delta_{M,-0}^{a_M^{(n,n)}} + \gamma_W \hat{\Theta}_{n,n-1} \Delta_{W,-0}^{a_W^{(n,n)}})$$
$$\Delta_{W,-0}^{b_W^{(n,n)}} = (I_{n-1} - \hat{\Theta}_{n,n-1}^2)^{-1}(\gamma_W \Delta_{W,-0}^{a_W^{(n,n)}} + \gamma_M \hat{\Theta}_{n,n-1} \Delta_{M,-0}^{a_M^{(n,n)}}).$$

It follows that

$$\begin{aligned}
&||\Delta_{W,-0}^{b_W^{(n,n)}} - \Delta_{M,-0}^{b_M^{(n,n)}}||_\infty = ||(I_{n-1} - \hat{\Theta}_{n,n-1}^2)^{-1}(I_{n-1} - \hat{\Theta}_{n,n-1})(\gamma_W \Delta_{W,-0}^{a_W^{(n,n)}} - \gamma_M \Delta_{M,-0}^{a_M^{(n,n)}})||_\infty \\
&= ||(I_{n-1} + \hat{\Theta}_{n,n-1})^{-1}(\gamma_W \Delta_{W,-0}^{a_W^{(n,n)}} - \gamma_M \Delta_{M,-0}^{a_M^{(n,n)}})||_\infty \\
&\leq ||(I_{n-1} + \hat{\Theta}_{n,n-1})^{-1}||_\infty(\gamma_W ||\Delta_{W,-0}^{a_W^{(n,n)}}||_\infty + \gamma_M ||\Delta_{M,-0}^{a_M^{(n,n)}}||_\infty).
\end{aligned}$$

But, $||(I_{n-1} + \hat{\Theta}_{n,n-1})^{-1}||_\infty = O(\sqrt{n})$. Indeed, from Lemma 2 (ii), we know:

$$\min_{i \in \{1,\ldots,n-1\}} (1 + \theta_{n,ii} - \sum_{j \neq i} \theta_{n,ij}) = \min_{i \in \{1,\ldots,n-1\}} \left(2\theta_{n,ii} + \frac{2}{n+1}\right).$$

Hence, Lemma 10 implies

$$\lim_{n\to\infty} \left(\frac{\sqrt{\pi n}}{2} \min_{i \in \{1,\ldots,n-1\}} (1 + \theta_{n,ii} - \sum_{j \neq i} \theta_{n,ij})\right) = 1.$$

Thus, by Lemma 8, $\limsup_{n\to\infty} \frac{2||(I_{n-1}+\hat{\Theta}_{n,n-1})^{-1}||_\infty}{\sqrt{\pi n}} \leq 1$, so that $||(I_{n-1} + \hat{\Theta}_{n,n-1})^{-1}||_\infty = O(\sqrt{n})$. Using $||\Delta_M^{a_M^{(n,n)}}||_\infty = O\left(\frac{1}{n}\right)$ and $||\Delta_W^{a_W^{(n,n)}}||_\infty = O\left(\frac{1}{n}\right)$, it follows that $||\Delta_{W,-0}^{b_W^{(n,n)}} - \Delta_{M,-0}^{b_M^{(n,n)}}||_\infty = O(\frac{1}{\sqrt{n}})$.

Part (ii): Analogous to the argument in the main text, the sum of all agents' ex-ante expected utilities is bounded by ex-ante expected aggregate match surplus:

$$\begin{aligned}
&E\left[\frac{1}{k}\sum_{i=1}^{k} M_{n_k-k+i:n_k} W_{i:k}\right] > \frac{n_k}{k}\int_0^{\overline{m}} U_M^{(n_k,k)}(m) f(m)dm + \int_0^{\overline{w}} U_W^{(n_k,k)}(w) g(w)dw \\
&= \gamma_M \int_0^{\overline{m}} \int_0^m \Psi^{(n_k,k)}(s) ds f(m) \frac{n_k}{k} dm \\
&+ E[b_M(M_{n_k-k+1:n_k})] + \gamma_W \int_0^{\overline{w}} \int_0^w \Phi^{(n_k,k)}(s) ds g(w) dw.
\end{aligned}$$

The strict inequality above holds because the investments of the men who fail to match are lost. Note that

$$\lim_{k\to\infty,k<n_k,\frac{k}{n_k}\to 1-r} E\left[\frac{1}{k}\sum_{i=1}^{k} M_{n_k-k+i:n_k} W_{i:k}\right] = S^{(r)},$$

$$\lim_{k\to\infty,k<n_k,\frac{k}{n_k}\to 1-r} \int_0^{\overline{m}}\int_0^{m} \Psi^{(n_k,k)}(s)dsf(m)\frac{n_k}{k}dm = R_M^{(r)},$$

$$\lim_{k\to\infty,k<n_k,\frac{k}{n_k}\to 1-r} \int_0^{\overline{w}}\int_0^{w} \Phi^{(n_k,k)}(s)dsg(w)dw = R_W^{(r)},$$

where the first identity follows from the law of large numbers for empirical distributions, and the second and the third identity follow from Lemma 7 and the Dominated Convergence Theorem. Moreover, $S^{(r)} = R_M^{(r)} + R_W^{(r)}$ (as $R_M^{(r)}$ and $R_W^{(r)}$ are the aggregate core utilities in the continuum model). Thus,

$$\limsup_{k\to\infty,k<n_k,\frac{k}{n_k}\to 1-r} E[b_M(M_{n_k-k+1:n_k})] \le \gamma_W R_M^{(r)} + \gamma_M R_W^{(r)}.$$

We show now that $\lim_{k\to\infty,k<n_k,\frac{k}{n_k}\to 1-r} E[b_M(M_{n_k-k+1:n_k})] = \gamma_W R_M^{(r)} + \gamma_M R_W^{(r)}$, i.e., the fraction of the difference between aggregate surplus and aggregate information rents that is dissipated converges to 0. Indeed, by the above observations, $E[b_M(M_{n_k-k+1:n_k})] = O(1)$ (in the considered limit). Moreover, for any $\varepsilon > 0$ and any sequence $j_k \ge \varepsilon k$, $E[b_M(M_{n_k-k-j_k:n_k})]$ converges to zero exponentially fast (e.g., because, by Lemma 6, the probability that $F_{n_k-k:n_k-1}(M_{n_k-k-j_k:n_k})$ is greater than ε declines exponentially). Thus, all investments that are wasted in expectation, $E[b_M(M_{1:n_k})],...,E[b_M(M_{n_k-k:n_k})]$ are at most of order 1, and, for any $\varepsilon > 0$, at most εk of these are not exponentially small. Consequently, the per-capita expected utility that is lost converges to zero. □

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