Asymmetric All-Pay Auctions with Two Types*

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Abstract

We characterize the unique equilibrium of an asymmetric all-pay auction with incomplete information. The two bidders' types are independently drawn from different two-point probability distributions: Types as well as probabilities differ among bidders.

Next we apply our results to information disclosure in contests. Recent research shows that bidders do not disclose any information if they can only decide between full disclosure or none. In contrast, we find that bidders always disclose some information if disclosure can be partial.

JEL Classification: D44, D72, D82.

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1   Introduction

This paper studies an asymmetric incomplete information all-pay auction. We assume that there are two bidders whose valuations for winning are drawn from asymmetric two-point distributions. Types as well as type-probabilities are asymmetric across bidders. The paper provides a complete and explicit characterization of the unique Nash equilibrium. Our model covers many qualitatively different cases – such as overlapping supports, disjoint supports, or bidders varying to different extent around the same mean valuation. The flexibility of our general two-types approach is reflected in the rich structure of equilibrium we obtain.

Since our model is both tractable and flexible, it is ideally suited for use within applications. We demonstrate this by studying the question of information sharing in all-pay auctions. We consider a model where contestants are either of a strong or a weak type with asymmetric probabilities. Initially, both types and type probabilities are private information. In a preliminary stage, bidders have the possibility to share some of their private information with their opponent by revealing their type probability. We can show that contestants are always willing to share some private information about their type with their opponent. This result is derived from our previous equilibrium analysis with little additional effort.

Only recently, Siegel (2009, 2010) has contributed significantly to the understanding of asymmetries in complete information all-pay auctions.[1] The incomplete information case is considerably less well-understood, mostly due to the fact that explicit solutions are typically difficult to obtain even for elementary examples. A better understanding is important for at least two reasons as outlined in the next two paragraphs.

From a theoretical point view, the fact that the revenue-equivalence theorem breaks down under asymmetries[2] necessitates a separate study of different auction types. This was begun by Maskin and Riley (1985) who characterize equilibria of first- and second-price winner-pay auctions for the two-bidder case with asymmetric two-point distributions. Our results can be seen as a direct continuation of this work.[3]

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[1] For the earlier literature, see among others Hillman and Samet (1987), Hillman and Riley (1989), Baye, Kovenock and de Vries (1996), and Clark and Riis (1998).

[2] See Fibich, Gavious and Sela (2004) for a recent reference.

[3] The setting of Maskin and Riley is more restrictive than the one we consider here since they assume that the weak types of both bidders have a valuation of zero.

From a more applied perspective, a need to study asymmetric all-pay auctions with incomplete information arises from the fact that all-pay auctions are very popular models of contests and related competitive situations.[4] Though theoretically highly interesting, most of the few existing results on all-pay auctions with asymmetries and incomplete information are too abstract for easy use within applications. As a consequence, the vast majority of the applied all-pay auctions literature studies ex ante asymmetries only for the tractable complete information case.[5] As we show in our application of information sharing, this is clearly not without loss of generality.

More specifically, the theoretical literature on asymmetric incomplete information all-pay auctions consists mainly of two contributions: For the case of continuously distributed valuations, Amann and Leininger (1996) analyze the two bidder case. They show existence and uniqueness of the equilibrium and characterize some of its properties, e.g. that at most one participant bids zero with positive probability. Parreiras and Rubinchik (2010) generalize the setting of Amann and Leininger to n bidders with asymmetric risk-attitudes. They characterize classes of examples where the equilibrium has properties that have been found in experimental studies, such as non-decreasing densities of bids and complete drop-out of some bidders. Both of these studies make an important theoretical contribution. Yet neither of them provides (or aims at providing) a flexible model that allows for an explicit characterization of equilibrium suitable for applications.[6] Similarly, their results are too abstract to allow for a non-technical analysis of the way the equilibrium reacts to small changes in the distribution of valuations. Our more specialized but explicit results overcome both of these limitations while still addressing a rich spectrum of asymmetries.

Technically, while we retain most of the tractability of the complete information case in our model, Siegel's (2009, 2010) approach of first identifying bidders' equilibrium

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[4]Konrad (2009) provides an overview with many references.

[5]We are aware of only two contributions in the applied all-pay auctions literature which consider ex ante asymmetries and private information simultaneously: Sahuguet (2006) analyzes all-pay auctions with bid-caps where bidders' valuations are drawn from uniform distributions with different upper bounds of the support. Münster (2009) studies a model of repeated all-pay auctions where asymmetries arise endogenously over time. He considers two-point distributions where the weak types have infinite bidding costs and where the strong types to have symmetric costs. Clearly, both models are considerably less flexible than the one we consider here.

[6]Amann and Leininger (1996) obtain explicit results for distributions of the form $F(v) = v^\alpha$. Yet this (one-parameter) class of distributions captures only a limited spectrum of asymmetries between bidders, and – probably for this reason – has not been picked up by applied research.

payoffs through abstract arguments and then constructing a corresponding equilibrium does not carry over to our setting. Instead, our approach is more similar to the one employed by Amann and Leininger (1992) in the case of continuously distributed valuations: We construct a family of equilibrium candidates downwards from the upper boundary of the supports, using the observation that bidders' payoffs must be balanced locally. We then identify the equilibrium by observing that the lower end of supports must be zero.

Our results on information sharing in contests nicely complement recent results by Kovenock, Morath and Münster (2010) on the same subject. They consider the case of continuously distributed types. They restrict the choice about information sharing to either full or no revelation of the contestants' private information. Obviously, this is for technical reasons – if bidders would reveal some (but not all) information, an asymmetric incomplete information all-pay auction would arise with the well-known technical difficulties. Our two-types approach allows to consider partial release of information without any technical complications. Kovenock, Morath and Münster (2010) show that in their setting bidders never prefer to share information. In contrast, we find that bidders are always willing to share some of their private information if partial disclosure is possible.

The paper proceeds as follows: In Section 2, we introduce the model and state some elementary observations. Section 3 characterizes the unique equilibrium. In Section 4, we apply our results to the problem of information sharing in contests. Section 5 concludes.

2   The Model

There is an all-pay auction with 2 bidders, who each have a valuation of 1 for winning. With probability $p_i \in (0, 1)$, bidder i is of strong type and has low marginal costs of exerting effort, c_i. With probability $1 - p_i$, he is of weak type and has high marginal costs of C_i. We assume $c_i < C_i < \infty$. The probability distributions are common knowledge. Each bidder knows his own type but not the type-realizations of his opponents. The bidder who exerts the highest effort e wins the auction. Ties are broken arbitrarily. The symmetric case of this model has been analyzed in Konrad (2004). Münster (2009) studies the case $c_i = c$ and $C_i = \infty$.[7]

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[7]To keep the notation simple, we thus assume that bidders and types differ in bidding costs but have the same valuation for winning. It is straightforward to extract from our results the

It follows easily that any equilibrium of this game must be in mixed strategies and that there are no atoms except possibly in zero just as in the complete information auction.[8] Thus each bidder's strategy can be represented by two distribution functions which are atomless (with zero as the possible exception): Bidder i utilizes F_i^c if he has low marginal costs c_i and F_i^C if he has high marginal costs C_i.

For fixed i, the supports of F_i^c and F_i^C must be disjoint (except possibly for boundary points and zero): Let $P_i(e)$ denote bidder i's probability of winning via an effort of e (given the other bidder's strategy). Note that $P_i(e)$ does not depend on i's type. Thus bidder i maximizes either

$$P_i(e) - c_i e \qquad \text{or} \qquad P_i(e) - C_i e.$$

At no e both maxima can be attained (except in zero when the first summands are zero), as the respective first order conditions are $P_i'(e) = c_i$ and $P_i'(e) = C_i$.[9]

Taking this argument one step further, we see that the strong type's payoff from exerting effort in the weak type's interval must be increasing in effort: The weak type earns constant expected payoffs on his interval. Since the strong type has lower marginal costs, increasing the effort must then be profitable for the strong type on these effort levels. Likewise, the weak type's payoff must be decreasing in effort on the strong type's interval. This implies that the strong type of a bidder must play strictly higher effort levels than the weak type in equilibrium. Note that the arguments given so far carry over to the case of n bidders and k types as well. We conclude by collecting our observations so far in the following corollary:

Corollary 1 *In any equilibrium, the union of all bidders' types' supports must be a bounded interval with lower boundary zero. At least one bidder never sets an atom on zero. Moreover, no bidder puts an atom on strictly positive effort levels. With two bidders, the union over all types' strategy supports must be the same for both bidders, i.e., both bidders mix down to zero, no bidder leaves gaps and both bidders mix up to the same highest bid. At a given effort level, a bidder competes either*

corresponding results for the case where bidders differ in both respects: Multiplying a type's valuation for winning and costs by the same constant θ leaves the type's equilibrium strategy unchanged and changes his equilibrium payoff by the factor θ.

[8] See Corollary 1.

[9] If one of bidder i's types has an open interval I in the support of his equilibrium strategy then $P_i(e)$ must be linear on I since costs are linear. This implies differentiability.

against the strong type of the other bidder, or against the weak type. The strong type of a bidder plays higher effort-levels than the weak type.

3 The Unique Equilibrium

We now characterize the unique equilibrium explicitly. Depending on the values of the parameters c_1, C_1, p_1 and c_2, C_2 and p_2, let us distinguish six different cases defined with the help of the following conditions:

$$p_1 c_1 > p_2 c_2, \tag{A1}$$

$$p_1 c_1 > p_2 c_2 + (1 - p_2) C_2, \tag{A2}$$

$$p_1 c_1 + (1 - p_1) C_1 > p_2 c_2 + (1 - p_2) C_2. \tag{A3}$$

Before we come to the intuition for these conditions, we introduce some notation by recalling from Corollary 1 the following properties any equilibrium must possess: A bidder i mixes in equilibrium on a gapless support. His strong type will mix up to some $\bar{e}$ which is the same for both bidders and down to some lower boundary which we call e_i. His weak type plays all the effort-levels between e_i and zero.

Let us first consider the effort interval on which the strong types of both bidders compete against each other. Which strong type has to play the more concentrated strategy, i.e. has to mix with a higher density? If (A1) is fulfilled,

$$p_1 c_1 > p_2 c_2, \tag{A1}$$

it must be the strong type of bidder 1 (which we call "strong 1" from now on).
Let us see why: Consider the strong type of bidder 2, strong 2. Increasing his effort slightly by ϵ inside the equilibrium support must not affect his payoffs. An effort increase can only pay out if strong 1 is active, which happens with probability p_1. (Winning over weak 1, the weak type of bidder 1, is certain.) Hence it has to hold:

$$p_1 (F_1^c(e_2 + \epsilon) - F_1^c(e_2)) = c_2 \epsilon, \tag{1}$$

where the left hand side denotes the expected additional gain by increasing the effort by ϵ. The right hand side denotes the additional cost. Taking ϵ to zero we

also obtain the density strong 1 has to play on the interval where strong 2 is active. It must be given by $f_1^c = \frac{c_2}{p_1}$. Yet then, $\frac{p_1}{c_2}$ denotes the length of the interval strong 1 would have to mix on if he always played against strong 2.

Let us compare the lengths of the intervals strong 1 and strong 2 would have to play on if they always played against the strong type of the other bidder:

$$\frac{p_1}{c_2} > \frac{p_2}{c_1}. \tag{A1'}$$

If (A1') holds, as both strong types mix up to the same upper boundary $\bar{e}$ (whose value we still have to determine), strong 1 has to mix down to lower effort levels than strong 2. So we see that strong 2 indeed always competes against the strong type of his competitor, in contrast to strong 1, who is left with some probability mass he has to "spend" elsewhere. Note that (A1') and (A1) are equivalent.

Let us for the rest of the section without loss of generality assume that (A1) is fulfilled. Hence we know that strong 1 has to mix over a larger interval than strong 2. We know he cannot mix over higher effort levels than strong 2. Hence he has to mix down to lower effort levels than strong 2. Will he even mix down to effort level zero? This depends on (A2):

$$p_1 c_1 > p_2 c_2 + (1 - p_2) C_2, \tag{A2}$$

If (A2) holds, strong 1 even has to mix over a larger interval than both strong 2 and weak 2 together. Note that (A2) resembles (A1) very much: The left hand side is identical in both conditions, as in both situations it is always the strong type of bidder 1 who is active. The left hand side of (A2) is analogous to the left hand side of (A1) if we reinterpret bidder 2 as a bidder 2′ who is always of strong type (p=1), but who has costs of $p_2 c_2 + (1 - p_2) C_2$. Then (A2) reads:

$$p_1 c_1 > 1 \cdot (p_2 c_2 + (1 - p_2) C_2), \tag{A2'}$$

Note that such a reinterpretation is valid here because bidder 2 always competes against strong 1, and never against weak 1 under (A2). Let us consider the corresponding picture of the shape of the equilibrium:

Strong 1 has to mix over such a long interval that there is no room for weak 1 to mix over any positive effort levels. Hence weak 1 has to put all his probability-mass on zero.
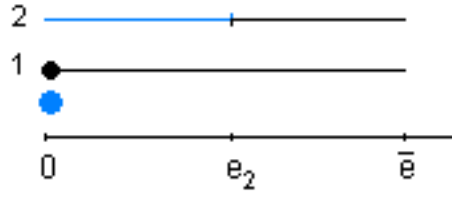
Figure 1: Supports of the bidders' strategies if (A1) and (A2) hold, strong types in black, weak types in blue.

Our third condition, Condition (A3),

$$p_1 c_1 + (1 - p_1)C_1 > p_2 c_2 + (1 - p_2)C_2, \tag{A3}$$

is relevant only if (A2) is not fulfilled, i.e. in the case where the weak types of both bidders exert positive efforts with some probability. Then it depends on (A3) which of the weak types puts an atom on zero. If (A3) is fulfilled, weak 1 has to play the atom in zero (and consequently earn zero profits):
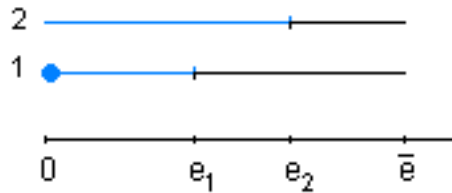

Figure 2: Supports of the bidders' strategies if (A1) holds, (A2) does not hold, but (A3) holds. Strong types in black, weak types in blue.

Conversely, if (A3) is violated, the weak type of bidder 2 puts an atom on zero and makes no profits.

Proposition 1 formally characterizes the unique equilibria for the three cases in which (A1) holds. The cases in which (A1) does not hold can of course be extracted from Proposition 1 by exchanging the roles of the indices 1 and 2. As one can see from the proposition, it is quite lengthy to state the equilibria explicitly. Yet note that the bidders just mix uniformly over the intervals specified before, with densities such that the opponent's active type would not gain or lose from marginally changing his effort level.
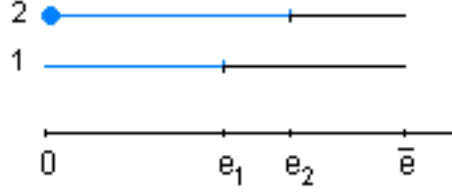
Figure 3: Supports of the bidders' strategies if (A1) holds, but (A2) and (A3) do not hold. Strong types in black, weak types in blue.

Proposition 1 *Consider the two bidder case and assume that (A1) holds. Then we have to distinguish three cases:*

1. *Assume (A2) holds. Define boundaries e_1, e_2, and $\bar{e}$ by $e_1 = 0$,*

$$e_2 = \frac{(1-p_2)C_2}{c_1 C_2} = \frac{1-p_2}{c_1}, \qquad \bar{e} = e_2 + \frac{p_2 c_2}{c_1 c_2} = \frac{1}{c_1}.$$

In the unique equilibrium, weak 1 places an atom of size 1 on 0. Strong 1 mixes over $(e_1, e_2]$ with constant density $\frac{C_2}{p_1}$ and over $(e_2, \bar{e}]$ with density $\frac{c_2}{p_1}$. Additionally, strong 1 places an atom of size

$$\frac{p_1 c_1 - p_2 c_2 - (1-p_2)C_2}{p_1 c_1}$$

on e_1. Weak 2 mixes over $(e_1, e_2]$ with density $\frac{c_1}{(1-p_2)}$. Strong 2 mixes over $(e_2, \bar{e}]$ with density $\frac{c_1}{p_2}$.

2. *Assume (A2) does not hold but (A3) does. Define boundaries e_1, e_2, and $\bar{e}$ by*

$$e_1 = \frac{p_2 c_2 + (1-p_2)C_2 - p_1 c_1}{C_1 C_2}, \qquad e_2 = e_1 + \frac{p_1 c_1 - p_2 c_2}{c_1 C_2}, \qquad \bar{e} = e_2 + \frac{p_2 c_2}{c_1 c_2}.$$

Then the unique equilibrium is given by the following strategies: Weak 1 mixes over $(0, e_1]$ with density $\frac{C_2}{1-p_1}$ and places an atom of size

$$\frac{p_1 c_1 + (1-p_1)C_1 - p_2 c_2 - (1-p_2)C_2}{(1-p_1)C_1}$$

on 0. Strong 1 mixes over $(e_1, e_2]$ with density $\frac{C_2}{p_1}$ and over $(e_2, \bar{e}]$ with density $\frac{c_2}{p_1}$. Weak 2 mixes over $(0, e_1]$ with density $\frac{C_1}{1-p_2}$ and over $(e_1, e_2]$ with density $\frac{c_1}{1-p_2}$. Strong 2 mixes over $(e_2, \bar{e}]$ with density $\frac{c_1}{p_2}$.

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3. *Assume (A2) and (A3) both do not hold. Define boundaries e_1, e_2, and $\bar{e}$ by*

$$e_1 = \frac{(1-p_1)C_1}{C_1 C_2}, \qquad e_2 = e_1 + \frac{p_1 c_1 - p_2 c_2}{c_1 C_2}, \qquad \bar{e} = e_2 + \frac{p_2 c_2}{c_1 c_2}.$$

Then the unique equilibrium is given by the following strategies: Weak 1 mixes over $(0, e_1]$ with density $\frac{C_2}{1-p_1}$. Strong 1 mixes over $(e_1, e_2]$ with density $\frac{C_2}{p_1}$ and over $(e_2, \bar{e}]$ with density $\frac{c_2}{p_1}$. Weak 2 mixes over $(0, e_1]$ with density $\frac{C_1}{1-p_2}$ and over $(e_1, e_2]$ with density $\frac{c_1}{1-p_2}$. Additionally, weak 2 places an atom of size

$$\frac{p_2 c_2 + (1-p_2)C_2 - p_1 c_1 - (1-p_1)C_1}{(1-p_2)C_2}$$

on 0. Strong 2 mixes over $(e_2, \bar{e}]$ with density $\frac{c_1}{p_2}$.

From Proposition 1 it is easy to calculate the expected equilibrium payoffs:

Corollary 2 *In the setting of Proposition 1, the payoff of strong i equals $1 - c_i \bar{e}$. The payoff of weak i equals A_{-i}, where A_{-i} is the probability that i's opponent exerts an effort of zero.*

Just like bidders in a complete information all-pay auction, the strong types earn the same expected payoffs if $c_1 = c_2$. Yet the same is not true for weak type bidders. Even if $C_1 = C_2$ their expected payoffs will generally differ: Due to the atom, one of them earns a positive expected payoff while the other obtains zero payoff.

To gain some more intuition, and since we work with this case in Section 4, we finish our analysis of the two bidder case with a closer look at the situation where asymmetries lie only in the probabilities, i.e. $c_1 = c_2 = c$ and $C_1 = C_2 = C$. Then, assumption (A1), i.e. $p_1 > p_2$, immediately implies that (A2) and (A3) must be violated. Hence we are then always in the third case of Proposition 1. Then we get the following simplified formulas for the payoffs:

Corollary 3 *Assume that in the setting of Proposition 1 it holds that $c_1 = c_2 = c$ and $C_1 = C_2 = C$. Then the atom of the opponent is given by*

$$A_{-i} = (p_i - \min(p_1, p_2))(1 - \frac{c}{C}).$$

The upper bound of supports $\bar{e}$ is given by

$$\bar{e} = \frac{\min(p_1, p_2)}{c} + \frac{1 - \min(p_1, p_2)}{C}.$$

Accordingly, the expected payoff of weak i is given by

$$\pi_i^w = (p_i - \min(p_1, p_2)) \left(1 - \frac{c}{C}\right).$$

The expected payoff of strong i

$$\pi_i^s = (1 - \min(p_1, p_2)) \left(1 - \frac{c}{C}\right).$$

Note that the corollary is written in a way that it holds regardless of whether (A1) is fulfilled or not.

4 Information Sharing in Contests

In this section we apply the analysis of Section 3 to the study of incomplete information contests where bidders have the opportunity to share some information about their type. Our aim in studying this problem is twofold: First, the issue is interesting in itself. Our results add an interesting new perspective to recent results of Kovenock, Morath and Münster (2010) (KMM in the following) as outlined below. Second, studying this problem allows us to demonstrate how easily our previous results can be applied to richer frameworks.

We consider the following setting: There are two bidders, both with a valuation of 1 for the object for sale in an all-pay auction.[10] Bidder i's marginal costs of exerting effort are c with probability p_i and C with probability $1 - p_i$ where $0 < c < C < \infty$. The probabilities p_i are independent random variables drawn from a distribution F on $[0, 1]$ with $E[p_i] = \mu$. Ex ante, the bidders only know F, c, and C. They know neither their realization of p_i nor the realization of their costs of bidding.[11] The timing is as follows:

1. The bidders decide whether to share information later in the game, at stage 3. The decision game is either modeled through simultaneous voting or through individual decisions as described below.

2. The bidders learn their realization of p_i. Hence, each bidder receives a more concrete estimate of his type.

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[10]Our results also hold when bidders are heterogeneous in valuations instead of effort costs.

[11]For the sake of brevity, we focus on ex ante decisions here. Yet all our results of this section carry over to the situation where bidders decide about sharing after they have learned their valuations.

3. Depending on the decision at stage 1, the realizations of the p_i become common knowledge or remain private information.

4. The bidders learn the realization of their types and the all-pay auction takes place.

We consider two different decision frameworks for information-sharing:

1. **Simultaneous Voting:** Both bidders simultaneously cast a vote for or against sharing information. If both bidders vote for sharing, valuations are revealed and a complete information all-pay auction takes place in the final stage. Otherwise, an incomplete information all-pay auction takes place.

2. **Independent Decisions:** The bidders independently make an ex ante commitment about sharing information or not. Depending on the bidders' decisions, either both valuations, or only one, or none become common knowledge before the auction takes place.

KMM also analyze an independent private values[12] all-pay auction with two bidders where bidders are ex ante uninformed about their valuations. Bidders decide whether they would like to share information. Depending on the bidders' sharing decision, which is modeled like in our setting, the bidders play either a complete information all-pay auction or an incomplete information all-pay auction. Thus unlike in our model, bidders do not have the possibility to partially disclose their private information.

KMM obtain the following results: For the simultaneous voting case, KMM show that the complete and incomplete information auctions yield the same payoffs for the bidders, implying that any choice of actions is a Nash equilibrium.[13] The loss in informational rents from disclosing is exactly off-set by the economic rents arising from the bidders' different strengths becoming common knowledge. For the case of independent decisions, KMM show that sharing information is strictly dominated.

We now analyze bidders' sharing decisions in our framework where partial disclosure is possible. It turns out that we obtain essentially the opposite of the results

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[12]The second part of KMM considers common value auctions. These will not be discussed here.

[13]To see this, recall that in a complete information all-pay auction the stronger bidder earns the difference in valuations while the other bidder earns zero payoffs. To see that the same is true for the incomplete information case, note that these payoffs are identical to those of an incomplete information second price auction. Thus by revenue equivalence these are also the payoffs of the incomplete information all-pay auction.

of KMM. We find that the trade-off between gaining economic rents and losing informational rents is not as simple as one might think: For any partial release of information, the gain in economic rents strictly dominates the loss in informational rents.

For the case where both bidders disclose their p_i, Corollary 3 provides the payoffs of the all-pay auction. It is easy to see that when bidder i does not share his realization of p_i, it is an equilibrium that both bidders still play the equilibrium of Corollary 3 but with μ instead of p_i.[14]

The following corollary - which is an immediate consequence of Corollary 3 - states the bidders' ex ante expected payoffs for the different decisions about sharing information.

Corollary 4 *Define $\theta = 1 - \frac{c}{C}$. Denote by $\pi_1(1r,\ 2n)$ bidder 1's ex ante expected payoff if bidder 1 reveals his p_1 and bidder 2 does not reveal p_2. $\pi_1(1r,\ 2r)$, $\pi_1(1n,\ 2r)$ and $\pi_1(1n,\ 2n)$ are defined analogously. The ex ante expected payoffs from the all-pay auction for the different disclosure decisions are as follows:*

1. *If both bidders decide to reveal their p_i, the ex ante expected payoff of bidder 1 is*

$$\pi_1(1r,\ 2r) = E[p_1(1 - \min(p_1, p_2)) + (1 - p_1)(p_1 - \min(p_1, p_2))]\theta.$$

2. *If bidder 1 decides to reveal but bidder 2 does not, the ex ante expected payoff of bidder 1 is*

$$\pi_1(1r,\ 2n) = E[p_1(1 - \min(p_1, \mu)) + (1 - p_1)(p_1 - \min(p_1, \mu))]\theta.$$

3. *If bidder 1 does not reveal but bidder 2 does, the ex ante expected payoff of bidder 1 is*

$$\pi_1(1n,\ 2r) = E[p_1(1 - \min(\mu, p_2)) + (1 - p_1)(\mu - \min(\mu, p_2))]\theta.$$

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[14]There are further equilibria where the bidders utilize their realizations of p_i as a randomizing device. These equilibria are however all payoff-equivalent to the one of Corollary 3. The reason for the payoff-equivalence is that - as during the auction the bidders know their own types - the private information about the p_i is useless unless it is shared.

4. If both bidders decide not to reveal, the ex ante expected payoff of bidder 1 is

$$\pi_1(\textit{1n, 2n}) = E[p_1(1 - \min(\mu, \mu)) + (1 - p_1)(\mu - \min(\mu, \mu))]\theta = \mu(1 - \mu)\theta.$$

Let us now consider the bidders' disclosure decisions in the simultaneous voting regime. Since each bidder can veto against information-sharing, it is always a weak Nash equilibrium that both bidders vote against disclosure. In the model of KMM, payoffs are the same regardless of disclosure decisions. Thus in their setting any vector of strategies is a weak Nash equilibrium. In our model we obtain instead:

Corollary 5 *Consider the simultaneous voting case. Assume that neither $p_i = \mu$ a.s. nor $p_i \in \{0, 1\}$ a.s.. Then $\pi_1(\textit{1r, 2r}) > \pi_1(\textit{1n, 2n})$. Thus in the only strict Nash equilibrium both bidders vote for information-sharing. This equilibrium is also strictly payoff-dominant.*

In the corollary we excluded two cases: If p_i is deterministic, disclosure transports no information. If p_i is always either zero or one, disclosure is fully revealing such that we are essentially in the setting of KMM. The corollary follows immediately from observing that (except in the two excluded cases)

$$\pi_1(\text{1r, 2r}) > E[p_1(1 - p_2)]\theta = \mu(1 - \mu)\theta = \pi_1(\text{1n, 2n}).$$

Let us now turn to the game with individual decisions on information sharing. Under this regime, KMM show that in their setting committing to reveal information is a strictly dominated action. Accordingly, the unique Nash equilibrium is that both bidders do not disclose. This is in contrast to our model with partial revelation:

Corollary 6 *Consider the individual decisions case. Assume that neither $p_i = \mu$ a.s. nor $p_i \in \{0, 1\}$ a.s.. Then it holds that $\pi_1(\textit{1r, 2n}) > \pi_2(\textit{1n, 2n})$. Thus, given that the opponent does not reveal, it is a strict best response for a bidder to reveal. Hence it is not a Nash equilibrium that both bidders withhold their private information.*

The corollary follows immediately from the observation that

$$\pi_1(\text{1r, 2n}) > E[p_1(1 - \min(p_1, \mu))]\theta > E[p_1(1 - \mu)]\theta = \pi_1(\text{1n, 2n}).$$

It depends on the distribution F whether a bidder prefers to reveal or not, given

that his opponent reveals. Yet in any case, some bidder will reveal at least with some probability in equilibrium.[15]

Combining our results and those of KMM shows that it will be difficult to settle the issue of information sharing in contests without a model that has both a sufficiently rich type-space and a sufficiently rich model of information revelation. This is a challenging direction for further research. In the situation with simultaneous voting, the result of KMM looks essentially like a boundary case of our result. We hence conjecture that the result of Corollary 5 is quite robust. The situation with individual decisions is more complex. KMM rightly point out that the two-types case is an extreme case concerning individual sharing decisions: With two types, bidders are indifferent between completely revealing and not revealing regardless of the opponent's behavior. This does not carry over to a state space with more than two types. Yet we have seen that *partial* sharing is a *strict* best response to an opponent who does not share. It seems highly intuitive that this strict advantage of sharing will not disappear instantly, e.g., whenever a third (possibly very unlikely) type is introduced. We thus conjecture that it will depend sensitively on the distribution of types and other model parameters whether bidders want to independently share information or not.

5 Conclusion

We have analyzed an asymmetric all-pay auction with incomplete information about the different bidders' types and their different type-probabilities. The assumption of two different types enabled us to carry out an explicit analysis of equilibrium in an asymmetric auction setting when information is incomplete.

With our results, one can study asymmetries in all areas in which all-pay auctions are popular models, such as lobbying, rent-seeking, R&D activities, or sport contests.[16] Our results may also serve as an easy-to-use tool for the analysis of richer models. For instance, we considered models with multiple stages, where asymmetries often arise naturally in the course of the game.

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[15]For example, if F is the uniform distribution on $[0, 1]$, there are three Nash Equilibria: two equilibria where one bidder reveals for sure while the other does not, and a symmetric mixed equilibrium.

[16]For an overview, see Konrad (2009).

A   Proofs

Proof of Corollary 1

We first show that bidders do not set atoms except possibly in zero: Assume a type of bidder i placed an atom in $e > 0$. If some type of the other bidder j was active right below e, then j would prefer to bid slightly more than e. If bidder j was inactive right below e, then i would prefer to shift his atom downwards a little. Next we observe that at most one bidder has types who set an atom on zero: Assume both bidders played zero with positive probability. Then at least one of them (depending on the tie-breaking rule) would prefer to shift his atom slightly upwards.[17] Next observe that both bidders must mix over the same support which is an interval: Assume a type of bidder i mixed over an interval I on which no type of bidder j is active. Then bidder i would prefer to shift his mass in the interval to its lower boundary. This argument also implies that both bidders must mix up to the same highest bid. Likewise, assume there is an interval I on which neither bidder is active but some bidders are active right above I. Then it would be profitable for such a bidder to deviate by moving mass from slightly above I into the interval. By the same argument, there cannot be an interval above zero on which no bidder is active. Thus bidders' supports must go down to zero. In the main text we have argued why different types of a bidder must mix over distinct supports and why stronger types mix over higher supports. $\qquad\square$

Proof of Proposition 1

The proof is structured as follows: We collect observations about the shape of the equilibrium until we know enough to calculate a unique equilibrium candidate. It is then easy to verify that this candidate is indeed an equilibrium.

From Corollary 1 we know that the supports form an interval $[0, \bar{e}]$ for some $\bar{e} > 0$. We also know that there are points e_1 and e_2 in this interval such that weak i mixes over $[0, e_i]$ and strong i mixes over $[e_i, \bar{e}]$. Additionally, the weak type of at most one of the bidders may place an atom on zero. (Only) if weak i puts all his mass on zero, strong i may play an atom in zero as well. Moreover, as argued in the main text, we explicitly know the densities chosen by the different types against different opponents: If on an interval I strong i and weak j are active, strong i mixes with density C_j/p_i and weak j mixes with density $c_i/(1 - p_j)$. Generally a

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[17]As can be seen from the explicit equilibrium given in Proposition 1, there are cases where both types of one bidder place an atom in zero.

type's density is always the quotient of the opponent type's marginal cost and his own type probability.

We can thus sequentially calculate an equilibrium candidate: Fix some value of $\bar{e}$. Let the strong types of both bidders mix down from $\bar{e}$ until one of them, say i, has used up all his probability mass. We call this point (known only in reference to $\bar{e}$) e_i. At this point, the weak type of bidder i comes in. Repeat this procedure downwards to the point e_0 where both types of one bidder have used up all their probability mass. The opponent must put his remaining probability mass on an atom in e_0 since both bidders' have to mix over the same support. Note that for any $\bar{e}$, this procedure necessarily produces unique values e_0, e_1 and e_2. Now recall that e_0 must equal zero in equilibrium. This uniquely determines the values of e_1, e_2 and $\bar{e}$. We thus find a unique equilibrium candidate. It is tedious but straightforward to verify that this sequential procedure leads to the candidate stated in the proposition and that it is indeed an equilibrium. $\square$

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