Games and Economic Behavior 92 (2015) 138–149

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Tie-breaks and bid-caps in all-pay auctions ☆

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ARTICLE INFO

Article history:
Received 25 April 2011
Available online 26 June 2015

JEL classification:
D72
D44

Keywords:
All-pay auctions
Contests
Bid-caps
Tie-breaking
Lobbying

ABSTRACT

We revisit the two bidder complete information all-pay auction with bid-caps introduced by Che and Gale (1998), dropping their assumption that tie-breaking must be symmetric. Any choice of tie-breaking rule leads to a different set of Nash equilibria. Compared to the optimal bid-cap of Che and Gale we obtain that in order to maximize the sum of bids, the designer prefers to set a less restrictive bid-cap combined with a tie-breaking rule which slightly favors the weaker bidder. Moreover, the designer is better off breaking ties deterministically in favor of the weak bidder than symmetrically.

1. Introduction

At the Olympic Games of 1896, the first Olympic Games of the Modern Era, two weightlifters impressed the Athenian audience particularly: the Scotsman Launceston Elliot and the Dane Viggo Jensen. Both lifted the same highest weight. The jury decided to solve the tie in favor of Jensen for he was considered to have the better style. Unfamiliar with this tie-breaking rule, the British delegates protested against the decision. This finally led to Elliot and Jensen obtaining the permission to try again for lifting higher weights. Yet both failed. In the end, Jensen was declared the champion.

Nowadays, it is neither style nor the energy of the own country's delegates that helps to win a tie in a weightlifting-contest. Instead, when a tie occurs, it is resolved in favor of the lighter athlete. Behind this is the idea that a lighter athlete, though in the same weight-class, probably has to exert more effort to lift the same weight than a heavier competitor.

Clearly, sports contests are more interesting if athletes display great efforts. For a designer, it is hence a natural objective to maximize the sum of efforts exerted by the contestants. Che and Gale (1998) show that handicaps can be an effective tool for raising aggregate effort levels in all-pay contests but they restrict their analysis to symmetric tie-breaking. We are going to allow the designer not only to set handicaps optimally, but also to choose the optimal tie-breaking rule.

In a complete-information all-pay auction without bid caps, the stronger bidder's advantage arises from his ability to win with certainty. He can secure a positive payoff by bidding just above the weak bidder's valuation. If a bid cap less than the weak bidder's valuation is imposed, this advantage disappears. Moreover, the advantage can be reversed if the tie-breaking is sufficiently biased towards the weak bidder, since the weak bidder can then guarantee himself a positive payoff by bidding the cap. In fact, every choice of tie-breaking rule leads to a different set of Nash equilibria.

We provide a complete characterization of the rich equilibrium structure. By choosing an appropriate combination of tie-breaking rule and bid-cap, the designer can enforce pure equilibria as well as mixed equilibria in which either of the

☆ I would like to thank Hans-Peter Grüner, Benny Moldovanu, Frank Rosar, Patrick Schmitz, Nikolaus Schweizer, Ron Siegel and Christian Westheide.
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http://dx.doi.org/10.1016/j.geb.2015.05.008
0899-8256/

bidders earns zero payoff. Compared to Che and Gale's optimal bid-cap under symmetric tie-breaking, we obtain that the designer optimally sets a less restrictive bid-cap combined with a tie-breaking rule which slightly favors the ex ante weaker bidder. Whereas the unique Nash equilibrium of the unrestricted all-pay auction is in mixed strategies, both of these policies force bidders to play a pure strategy equilibrium. Both bidders bid the bid-cap. The optimal policy exploits the fact that the weaker bidder is willing to bid more if tie-breaking is biased in his favor. If this bias is not too large this does not deter the stronger bidder from competing.

In many real-world settings, ties are broken either in favor of one bidder or 50 : 50. Therefore, we also consider the designer's problem if he is restricted to choosing between symmetric and deterministic tie-breaking rules. Even under this restriction, the designer can do better than in the optimal policy of Che and Gale (1998). In the optimum, he sets a bid-cap which is just small enough to influence equilibrium behavior and always breaks ties in favor of the weak bidder. Superficially, this policy seems like a minimal intervention into the game but it has important consequences. In an unrestricted all-pay auction, there is a mixed equilibrium in which the weak bidder stays out with a positive probability. The designer's policy gives rise to an equilibrium in which the weak bidder makes a preemptive bid (by bidding at the bid-cap) with the same probability with which he would stay out in the unrestricted auction.

Aggregate bids vary across policies as soon as asymmetries are present. Differences in aggregate bids increase as asymmetries grow larger. In the limit, the policy of Che and Gale outperforms the standard unrestricted all-pay auction by a factor of two. Solving the tie in favor of the weak bidder combined with the optimal bid-cap leads to an increase by a factor of three. If a tie-breaking slightly in favor of the weak bidder is possible, the optimal policy outperforms the standard auction by a factor of four.

Related literature In the vast literature on all-pay auctions,[1] tie-breaking rules have received comparatively little attention. Indeed, in many all-pay auction games, the set of Nash equilibria is invariant to the choice of tie-breaking rule. An example is a standard complete information all-pay auction in which at least two bidders have positive valuations for the object for sale.[2] In other related games, the choice of tie-breaking rule is a necessity since a Nash equilibrium exists only for certain tie-breaking rules. Consider for instance a two-player complete information all-pay auctions in which bidders have valuations $v_1 > 0$ and $v_2 = 0$. Then a Nash equilibrium (in which both bidders bid zero) exists only if tie-breaking always favors bidder 1.

In contrast, in all-pay auctions with binding bid-caps the choice of tie-breaking rule is decisive since in equilibrium both bidders play the bid-cap with positive probability. Yet in the literature only the case of symmetric tie-breaking has been considered. This concerns both the complete information case studied by Che and Gale (1998), Persico and Sahuguet (2006)[3] and Hart (2014),[4] and the incomplete information case studied by Gavious et al. (2003) and Sahuguet (2006). See Che and Gale (1998) for a discussion of the relation to policies other than bid-caps such as minimum-bid requirements.

Outline The paper is structured as follows. Section 2 introduces the model. Section 3 characterizes the bidders' equilibrium behavior for all combinations of bid-caps and tie-breaking rules. Section 4 analyzes the designer's optimization problem. First we allow for arbitrary tie-breaking rules, then we focus on tie-breaking rules that are either deterministic or symmetric. Section 5 discusses some extensions and implications of our analysis. All proofs are in Appendix A.

2. The model

We consider a complete information all-pay auction with two bidders 1 and 2 with positive valuations v_1 and v_2 for winning. Throughout we assume $v_1 > v_2$. Each bidder is restricted to choose his bid b from the interval $[0, m]$ at a cost of b. If a bidder submits the strictly highest bid, he wins. If both bidders submit the same bid, bidder 1 wins with probability $\alpha \in [0, 1]$, otherwise bidder 2 wins. We assume that before the auction takes place a designer chooses α and the handicap-level m in order to maximize the sum of bids. This is the setting of Che and Gale (1998) with the only difference that they restrict their analysis to symmetric tie-breaking.

If the designer does not impose a bid-cap, i.e. $m = \infty$, we are back to the standard complete information all-pay auction. In its unique equilibrium, both bidders mix uniformly over $[0, v_2)$.[5] Moreover, bidder 2 bids 0 with probability $1 - \frac{v_2}{v_1}$. For $m > v_2$, this set of strategies remains the unique equilibrium. The case $m = 0$ is trivial. Thus we assume in the following that the designer chooses (α, m) from the set

$$C = [0, 1] \times (0, v_2].$$

We denote by C_G the subset analyzed by Che and Gale (1998), $C_G = \left\{\frac{1}{2}\right\} \times (0, v_2]$.

[1] See Konrad (2009) for an overview.

[2] This has been shown, among others and in increasing generality, by Hillman and Samet (1987), Hillman and Riley (1989), Baye et al. (1996), and Siegel (2009). A parallel result holds for the incomplete information case studied first by Weber (1985) and Hillman and Riley (1989).

[3] Persico and Sahuguet (2006) embed the model of Che and Gale (1998) into a model of electoral competition in which parties try to attract heterogeneous voters. In their setting, the symmetric tie-breaking is implemented via the assumption that undecided voters toss a fair coin.

[4] Hart (2014) departs from the symmetry in Che and Gale's setting via analyzing asymmetric bid-caps under symmetric tie-breaking. This is similar to considering a tie-breaking always in favor of the less restricted bidder and implementing the more rigid cap for both bidders.

[5] See, e.g., Hillman and Riley (1989).

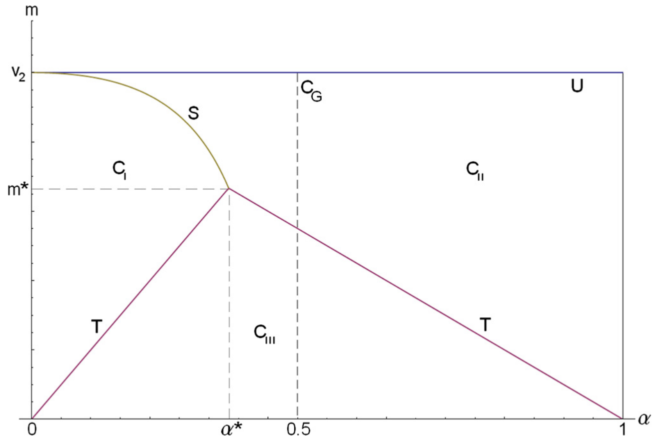

Fig. 1. Partition of the designer's choice set C.

3. All equilibria

In this section, we analyze the bidding game for fixed $(\alpha, m) \in C$. We characterize all Nash equilibria of this game. Five parameters suffice to specify each of these. We describe equilibrium strategies in terms of the associated cumulative distribution functions F_1 and F_2.

Proposition 1. *Fix $(\alpha, m) \in C$. If two distribution functions F_1 and F_2 form an equilibrium of the bidding game, then there exist numbers $c_1, d_1, c_2, d_2 \in [0, 1]$ and $\bar{b} \in [0, m]$ such that*

$$F_i(b) = c_i \, 1_{\{b \geq 0\}} + \frac{\min(b, \bar{b})}{v_j} + d_i \, 1_{\{b \geq m\}} \tag{1}$$

for $i, j \in \{1, 2\}$ and $j \neq i$. Moreover, $c_i > 0$ implies $c_j = 0$.

Thus, equilibrium strategies are characterized by possible atoms of mass c_i and d_i at 0 and at m and an interval $[0, \bar{b}]$ over which bidders mix uniformly with density inversely proportional to their opponent's valuation. At most one bidder places an atom at 0. The arguments which rule out atoms in the interior $(0, m)$ of the bid-space are similar as in a standard all-pay auction. Yet, unlike in a standard all-pay auction, we cannot rule out simultaneous atoms at m since a profitable deviation by bidding slightly above m is not available. Proposition 1 leads to the following classification of possible types of equilibria.

Corollary 1. *Fix $(\alpha, m) \in C$.*

(i) If there exists a pure equilibrium, it is given by both bidders bidding m.
(ii) In any equilibrium which is not pure, one bidder earns zero expected payoff.
(iii) If the pair of pure strategies in which both bidders bid m leads to strictly positive expected payoffs for both bidders, then this pair of strategies is the unique equilibrium of the game.

There can thus be pure equilibria in which both bidders bid m, and mixed equilibria in which at least one bidder earns a payoff of zero. In order to characterize the regions in C for which either is the case we need to define a decomposition of C into subsets. The decomposition is illustrated in Fig. 1.

Definition 1. Define critical levels of m by

$$m_1(\alpha) = \min(\alpha v_1, (1 - \alpha) v_2) \quad \text{for } \alpha \in [0, 1]$$

and

$$m_2(\alpha) = v_2 - \frac{\alpha^2}{1 - 2\alpha}(v_1 - v_2) \quad \text{for } \alpha \in \left[0, \alpha^*\right]$$

where $\alpha^* = v_2/(v_1 + v_2)$ denotes the maximizer of m_1. Furthermore, define

$$C_I = \left\{(\alpha, m) \in C \,\middle|\, \alpha \in \left[0, \alpha^*\right) \text{ and } m_2(\alpha) > m > m_1(\alpha)\right\},$$

as well as

$$C_{II} = \left\{(\alpha, m) \in C \left| \alpha \in \left[0, \alpha^*\right) \text{ and } v_2 > m > m_2(\alpha)\right.\right\} \cup \left\{(\alpha, m) \in C \left| \alpha \in \left[\alpha^*, 1\right] \text{ and } v_2 > m > m_1(\alpha)\right.\right\},$$

and

$$C_{III} = \{(\alpha, m) \in C \,|m < m_1(\alpha)\}.$$

Thus, $m_1(\alpha)$ separates the set C_{III} from C_I and C_{II}, while $m_2(\alpha)$ separates C_I and C_{II}. Denote by S the segment of C separating C_I and C_{II},

$$S = \left\{(\alpha, m) \in C \left| \alpha \in \left(0, \alpha^*\right) \text{ and } m = m_2(\alpha)\right.\right\},$$

by T the segment of C separating C_I and C_{II} from C_{III},

$$T = \{(\alpha, m) \in C \,|\alpha \in [0, 1] \text{ and } m = m_1(\alpha)\},$$

and by U the upper boundary, $U = [0, 1] \times \{v_2\}$.

Thus, $C = C_I \cup C_{II} \cup C_{III} \cup S \cup T \cup U$ and all sets in this conjunction are disjoint. In our characterization of equilibria we proceed in two steps: Proposition 2 proves existence and uniqueness of equilibrium for $(\alpha, m) \in C_I \cup C_{II} \cup C_{III}$. On C_I and C_{II} we find mixed equilibria in which, respectively, bidder 1 and bidder 2 bid zero with positive probability. On C_{III} both bidders bid m with probability 1 in equilibrium. The boundary cases $S \cup T \cup U$ are treated separately in Proposition 3.

Proposition 2. *For $(\alpha, m) \in C_I \cup C_{II} \cup C_{III}$ the all-pay auction with bid-cap has a unique equilibrium. Denote by π_i the equilibrium payoff of bidder i. Denote by $(c_1, d_1, c_2, d_2, \bar{b})$ the parameters characterizing bidding strategies of the form (1).*

(i) For $(\alpha, m) \in C_I$, expected payoffs are $\pi_1 = 0$ and

$$\pi_2 = v_2 - \left(\frac{\alpha^2}{(1-\alpha)^2}\, v_1 + \left(1 - \frac{\alpha^2}{(1-\alpha)^2}\right) m\right).$$

The equilibrium bidding strategies are given by

$$(c_1, d_1, c_2, d_2, \bar{b}) = \left(\frac{\pi_2}{v_2}, \frac{\alpha(v_1 - m)}{(1-\alpha)^2\, v_2}, 0, \frac{v_1 - m}{(1-\alpha) v_1}, \frac{m - \alpha v_1}{1-\alpha}\right). \tag{2}$$

(ii) For $(\alpha, m) \in C_{II}$, expected payoffs are $\pi_2 = 0$ and

$$\pi_1 = v_1 - \left(\frac{(1-\alpha)^2}{\alpha^2}\, v_2 + \left(1 - \frac{(1-\alpha)^2}{\alpha^2}\right) m\right).$$

The equilibrium bidding strategies are given by

$$(c_1, d_1, c_2, d_2, \bar{b}) = \left(0, \frac{v_2 - m}{\alpha v_2}, \frac{\pi_1}{v_1}, \frac{(1-\alpha)(v_2 - m)}{\alpha^2\, v_1}, \frac{m - (1-\alpha) v_2}{\alpha}\right). \tag{3}$$

(iii) For $(\alpha, m) \in C_{III}$, expected payoffs are $\pi_1 = \alpha v_1 - m$ and $\pi_2 = (1-\alpha) v_2 - m$. The equilibrium bidding strategies are given by

$$(c_1, d_1, c_2, d_2, \bar{b}) = (0, 1, 0, 1, 0).$$

Proposition 3 characterizes the sets of equilibria in the boundary cases $S \cup T \cup U$.

Proposition 3.

(i) For $(\alpha, m) \in S$ the unique equilibrium is given by

$$(c_1, d_1, c_2, d_2, \bar{b}) = \left(0, \frac{\alpha(v_1 - v_2)}{(1-2\alpha) v_2}, 0, \frac{(1-\alpha)(v_1 - v_2)}{(1-2\alpha) v_1}, \frac{(1-\alpha) v_2 - \alpha v_1}{1-2\alpha}\right).$$

(ii) For $(\alpha, m) \in T$ with $\alpha < \alpha^*$ the set of all equilibria is given by all strategy pairs of the form (1) with

$$(c_1, d_1, c_2, d_2, \bar{b}) = (\theta,\ 1-\theta,\ 0,\ 1,\ 0) \quad \text{where} \quad \theta \in \left[0, 1 - \frac{m}{(1-\alpha)\, v_2}\right].$$

(iii) For $(\alpha^*, m^*) \in T$ where $m^* = m_1(\alpha^*)$ the unique equilibrium is given by

$$(c_1, d_1, c_2, d_2, \bar{b}) = (0,\ 1,\ 0,\ 1,\ 0)\,.$$

(iv) For $(\alpha, m) \in T$ with $\alpha > \alpha^*$ the set of all equilibria is given by all strategy pairs of the form (1) with

$$(c_1, d_1, c_2, d_2, \bar{b}) = (0,\ 1,\ \theta,\ 1-\theta,\ 0) \quad \textit{where} \quad \theta \in \left[0, 1 - \frac{m}{\alpha\, v_1}\right].$$

(v) For $(\alpha, m) \in U \setminus \{(0, v_2)\}$, the unique equilibrium is given by

$$(c_1, d_1, c_2, d_2, \bar{b}) = \left(0,\ 0,\ 1 - \frac{v_2}{v_1},\ 0,\ v_2\right).$$

(vi) For $(\alpha, m) = (0, v_2) \in U$, the set of all equilibria is given by all strategy pairs of the form (1) with

$$(c_1, d_1, c_2, d_2, \bar{b}) = \left(0,\ 0,\ \theta,\ 1 - \frac{v_2}{v_1} - \theta,\ v_2\right) \quad \textit{where} \quad \theta \in \left[0, 1 - \frac{v_2}{v_1}\right].$$

Let us close this section with some observations about the structure of equilibria.

Remark 1.

(i) Compared to the case $\alpha = \frac{1}{2}$ considered by Che and Gale (1998), we observe a considerable variety in possible equilibrium outcomes. For any m and $\alpha \in \{0, 1\}$, there are equilibria with payoffs given by $(\pi_1, \pi_2) = (0, v_2 - m)$ and $(\pi_1, \pi_2) = (v_1 - m, 0)$. A broad range of parameters yields mixed equilibria in which the weaker bidder 2 earns a positive payoff. Equilibria of this type do not exist for $\alpha = \frac{1}{2}$.

(ii) In light of Proposition 2, Che and Gale's observation that mixed equilibrium payoffs are not influenced by m is an artifact of the case $\alpha = \frac{1}{2}$. Within C_{II}, π_1 is increasing in m for $\alpha > \frac{1}{2}$ and decreasing for $\alpha < \frac{1}{2}$.

(iii) Comparing Cases (v) and (vi) of Proposition 3, we see that for $m = v_2$ the equilibrium of the unconstrained complete information all-pay auction remains an equilibrium. Always breaking ties in favor of the weaker bidder does however destroy its uniqueness.

Equilibrium non-uniqueness in Proposition 3 arises because one bidder has some probability mass which he can place either in 0 or in m. The other bidder's equilibrium strategy is identical across all equilibria. The next corollary shows that, in the limit $\varepsilon \downarrow 0$, the unique equilibrium for $m - \varepsilon$ is arbitrarily close to the equilibrium in which the indifferent bidder never bids zero. These are the equilibria with $\theta = 0$ in Proposition 3.

Corollary 2. *Let $(\alpha, m) \in T \cup \{(0, v_2)\}$ and let $\varepsilon > 0$ be sufficiently small to guarantee $(\alpha, m - \varepsilon) \in C_I \cup C_{III}$. Define $p(\varepsilon) = (c_1(\varepsilon), d_1(\varepsilon), c_2(\varepsilon), d_2(\varepsilon), \bar{b}(\varepsilon))$ as the parameter vector which characterizes the unique equilibrium of the game with tie-breaking probability α and cap $m - \varepsilon$ from Proposition 2 (i) and (iii). For $(\alpha, m) \in T$, we have*

$$\lim_{\varepsilon \downarrow 0} p(\varepsilon) = (0, 1, 0, 1, 0)$$

and for $(\alpha, m) = (0, v_2)$, we have

$$\lim_{\varepsilon \downarrow 0} p(\varepsilon) = \left(0, 0, 0, 1 - \frac{v_2}{v_1}, v_2\right).$$

The limits thus correspond to the case $\theta = 0$ in Proposition 3 (ii, iv, vi).

The corollary is a direct consequence of Propositions 2 and 3. Convergence of the parameter vectors implies convergence of expected equilibrium bids, see Formula (4) below. In Section 4, we consider a designer who maximizes the expected sum of bids. In all cases of equilibrium multiplicity, the equilibrium with $\theta = 0$ yields the highest expected sum of bids. Corollary 2 shows that the designer can enforce a unique equilibrium arbitrarily close to the equilibrium with $\theta = 0$ by marginally decreasing the bid-cap. We thus follow Che and Gale (1998) and always solve equilibrium multiplicity in favor of the equilibrium with the highest expected sum of bids, $\theta = 0$.

4. The designer's problem

We now analyze the designer's problem of maximizing the expected sum of bids through the choice of α and m. The designer's objective is thus given by maximizing

$$\sigma(\alpha, m) = \sum_{i=1}^{2} \left((1 - c_i - d_i) \frac{\bar{b}}{2} + d_i m \right) \tag{4}$$

where the values of c_i, d_i and $\bar{b}$ corresponding to (α, m) are given in Propositions 2 and 3.[6] For $(\alpha, m) \in C_{III} \cup T$ we have $\sigma(\alpha, m) = 2m$. Within that triangle it is thus optimal to choose m as large as possible. The optimum is given by the tip of the triangle (α^*, m^*) with α^* defined in Definition 1 and $m^* = m_1(\alpha^*)$. The next lemma characterizes the monotonicity behavior of σ on $C_I \cup C_{II}$.

Lemma 1. *For $(\alpha, m) \in C_I$ and for $(\alpha, m) \in C_{II}$ with $\alpha > \frac{1}{2}$, $\sigma(\alpha, m)$ is increasing in m. For $(\alpha, m) \in C_{II}$ with $\alpha < \frac{1}{2}$, $\sigma(\alpha, m)$ is decreasing in m.*

Thus imposing a stricter bid-cap lowers the sum of bids in the region C_I and in the part of C_{II} which lies to the right of C_G. For intermediate values of α, lowering the bid-cap has a positive effect on the sum of bids. Intuitively, in the former case a smaller bid cap increases an existing advantage of one bidder and thus weakens competition. In the latter, intermediate case, decreasing the bid cap strengthens competition since it alleviates ex ante differences coming from the differing valuations. Proposition 4, the main result of this section, shows that (α^*, m^*) is the globally optimal policy.

Proposition 4. *The unique optimal policy for $\sup_{(\alpha,m)\in C} \sigma(\alpha, m)$ is given by $P^* = (\alpha^*, m^*)$. The resulting optimal expected sum of bids is $\sigma^*(v_1, v_2) = \frac{2v_1 v_2}{v_1 + v_2}$.*

Compared to Che and Gale's optimal policy P^{CG} under the constraint $\alpha = \frac{1}{2}$, $P^{CG} = (\alpha^{CG}, m^{CG}) = (\frac{1}{2}, \frac{v_2}{2})$, the designer thus imposes a higher bid-cap and a tie-breaking rule which favors the weaker bidder.

The next result shows that the designer can do better than the Che–Gale policy even if he is restricted to simple tie-breaking rules. Simple tie-breaking rules are those which break ties either symmetrically or always in favor of the same bidder, $\alpha \in \{0, \frac{1}{2}, 1\}$.

Proposition 5. *Define $C_R = \left\{0, \frac{1}{2}, 1\right\} \times (0, v_2]$. The unique optimal policy for $\sup_{(\alpha,m)\in C_R} \sigma(\alpha, m)$ is given by $P^R = (\alpha^R, m^R) = (0, v_2)$. The resulting optimal expected sum of bids is $\sigma^R(v_1, v_2) = \frac{3}{2} v_2 - \frac{1}{2} \frac{v_2^2}{v_1}$.*

In the optimal simple policy, ties are always broken in favor of the weaker bidder. Further, the policy imposes a bid-cap equal to the weaker bidder's valuation. The resulting game is almost the standard complete information all-pay auction. Yet instead of sometimes bidding zero, the weak bidder sometimes submits a preemptive bid, see Proposition 3 (vi) with $\theta = 0$.[7]

We close this section with a quantitative comparison of the policies P^*, P^R and P^{CG}. We also include a policy $P^0 = (\alpha^0, m^0) = (\frac{1}{2}, v_2)$ which implements the equilibrium of the standard complete information all-pay auction. The sums of bids associated with P^{CG} and P^0 are given by $\sigma^{CG}(v_1, v_2) = v_2$ and $\sigma^0(v_1, v_2) = \frac{v_2}{2} + \frac{1}{2} \frac{v_2^2}{v_1}$. Considering the limits $v_1 \uparrow \infty$ and $v_1 \downarrow v_2$ we obtain the following.

Corollary 3. *Fix $v_2 > 0$ and define*

$$\Sigma(v_1) = \left(\sigma^*(v_1, v_2),\ \sigma^R(v_1, v_2),\ \sigma^{CG}(v_1, v_2),\ \sigma^0(v_1, v_2) \right).$$

The limiting behavior of the sums of bids is given by $\lim_{v_1 \uparrow \infty} \Sigma(v_1) = \left(2,\ \frac{3}{2},\ 1,\ \frac{1}{2}\right) \cdot v_2$ and $\lim_{v_1 \downarrow v_2} \Sigma(v_1) = (1,\ 1,\ 1,\ 1) \cdot v_2$.

Notice that policies P^* and P^R manage to exploit large values of v_1: $\sigma^*(v_1, v_2)$ and $\sigma^R(v_1, v_2)$ are increasing in v_1 for fixed v_2. In contrast, $\sigma^{CG}(v_1, v_2)$ is independent of v_1 and $\sigma^0(v_1, v_2)$ is decreasing in v_1. This monotonicity behavior,

[6] Our selection of equilibria guarantees that $\sigma(\alpha, m)$ is upper-hemicontinuous. A different selection criterion would not affect the global optimization in Proposition 4 since equilibrium is unique at the optimizer. In Proposition 5, the supremum would remain identical but it would not necessarily be attained. Almost-optimal strategies with marginally smaller m would however survive. The latter also applies to the optimization for $\alpha = \frac{1}{2}$ in Che and Gale (1998).

[7] Hart (2014) shows that essentially the same pair of equilibrium strategies as under P^R also arises in the optimum when the designer is restricted to $\alpha = \frac{1}{2}$ but can implement asymmetric bid-caps.

together with Corollary 3, implies that the optimal policy P^* is at most twice as good as the policy of Che and Gale and at most four times as good as the unrestricted all-pay auction.

5. Discussion

Reducing the sum of bids Che and Gale (1998)'s initial motivation for considering bid-caps in political lobbying is that bid-caps are often imposed in an attempt to reduce lobbying efforts. They find that contrary to this goal, bid-caps may increase lobbying.

Our analysis implies that this counter-intuitive effect is reversed if ties are broken suitably, e.g. in favor of the strong bidder. One can show that for a fixed bid-cap m setting $\alpha = 1$ minimizes the expected sum of bids. For any $m \in (0, v_2)$ this leads to a reduction in the sum of bids. In fact, coupled with tie-breaking in favor of the stronger bidder, bid-caps have just the effect which one might naively expect from a bid-cap: Compared to the unrestricted auction, bidding on $(0, m)$ is unaffected. The stronger bidder moves mass from (m, v_2) into an atom at m. The weaker bidder increases his atom at 0 by the mass he previously had in (m, v_2).

Maximizing the winning bid In some applications, the designer may be interested in maximizing the expected winning bid instead of the expected sum of bids. In such a situation, the designer can improve upon the unrestricted all-pay auction policy P^0 by setting a bid-cap and a suitable tie-breaking rule: Policy P^R induces higher bids from the weaker bidder without having an impact on the stronger bidder's behavior. Thus the designer can always improve upon the unrestricted all-pay auction. In fact, one can show that P^R is the optimal policy with regard to maximization of the expected winning bid.

The symmetric case If $v_1 = v_2 = v$, symmetric tie-breaking levels the playing field completely. Therefore, no asymmetric tie-breaking or bid-cap can increase the expected sum of bids. The next proposition characterizes all equilibria of the symmetric case.

Proposition 6. *Suppose $v_1 = v_2 = v$. Define m_1, T and C_{III} as in Definition 1.*

(i) For $(\alpha, m) \in C \setminus (T \cup C_{III})$ with $\alpha \le \frac{1}{2}$, there exists a unique equilibrium. Expected equilibrium payoffs are $\pi_1 = 0$ and

$$\pi_2 = \left(1 - \frac{\alpha^2}{(1-\alpha)^2}\right)(v - m).$$

The equilibrium bidding strategies are given by

$$(c_1, d_1, c_2, d_2, \bar{b}) = \left(\frac{\pi_2}{v}, \frac{\alpha(v-m)}{(1-\alpha)^2\, v}, 0, \frac{v-m}{(1-\alpha)v}, \frac{m-\alpha v}{1-\alpha}\right). \tag{5}$$

(ii) For $(\alpha, m) \in C \setminus (T \cup C_{III})$ with $\alpha > \frac{1}{2}$, there exists a unique equilibrium. Expected equilibrium payoffs are $\pi_2 = 0$ and

$$\pi_1 = \left(1 - \frac{(1-\alpha)^2}{\alpha^2}\right)(v - m).$$

The equilibrium bidding strategies are given by

$$(c_1, d_1, c_2, d_2, \bar{b}) = \left(0, \frac{v-m}{\alpha v}, \frac{\pi_1}{v}, \frac{(1-\alpha)(v-m)}{\alpha^2\, v}, \frac{m-(1-\alpha)v}{\alpha}\right). \tag{6}$$

(iii) For $(\alpha, m) \in T \cup C_{III}$, all equilibria are as in Proposition 2 (iii) and Proposition 3 (ii–iv).

The structure of equilibria resembles the asymmetric case. There are still three regions, one with a pure equilibrium and two with mixed equilibria in which either of the bidders earns a positive payoff. The pure equilibrium which maximizes the sum of bids still arises for $(\alpha^*, m^*) = (\frac{1}{2}, \frac{v}{2})$. P^* and P^{CG} thus coincide in the symmetric case. The expected sum of bids v is the same as in an unrestricted complete information all-pay auction. The main qualitative difference to the asymmetric case is that the boundary S between the two mixed cases has a different shape: It no longer connects (α^*, m^*) with one of the corners. Instead, it is symmetric and connects $(\alpha^*, m^*) = (\frac{1}{2}, \frac{v}{2})$ with $(\frac{1}{2}, v)$.

Appendix A. Proofs

Throughout the proofs we sometimes refer to the two bidders as i and j, implying that a statement holds for $(i, j) = (1, 2)$ and $(j, i) = (1, 2)$. We define $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$.

Proof of Proposition 1. The proof proceeds in three steps: In the first step, we show that the functions F_i are continuous over $(0, m)$. This implies that only 0 and m can be played with positive probability in equilibrium. The second step shows that if equilibrium strategies contain mixing over a subset of $[0, m]$, this subset must be an interval which goes down to zero and is identical for both bidders. In the third step, we pin down the behavior of F_i over $(0, \bar{b})$. Collecting the observations of Steps 1–3 allows to conclude the proof.

Step 1: *F_1 and F_2 are continuous over $(0, m)$. Moreover, we have $F_1(0) = 0$ or $F_2(0) = 0$.*

Recall that as cumulative distribution functions F_1 and F_2 are right-continuous, (weakly) increasing and bounded. In particular, the functions F_i possess left limits, i.e. $F_i(b^-) := \lim_{x \uparrow b} F_i(x)$ is well-defined. We define $F_i(0^-) := 0$.

To prove continuity we first show that F_1 and F_2 cannot both have an atom at some $b \in [0, m)$. The proof is by contradiction. Suppose there exists $b \in [0, m)$ with $F_i(b) > F_i(b^-) > 0$ for $i = 1, 2$. This entails that both bidders earn their equilibrium payoff when bidding b. The equilibrium payoff of bidder 1 is thus

$$v_1(\alpha(F_2(b) - F_2(b^-)) + F_2(b^-)) - b. \tag{7}$$

Now consider bidder 1's payoff from playing $b + \varepsilon$ for some $\varepsilon > 0$ with the property that F_2 is continuous at $b + \varepsilon$:

$$v_1 F_2(b + \varepsilon) - b - \varepsilon. \tag{8}$$

Since F_2 is monotonic, any neighborhood of b contains infinitely many points where F_2 is continuous. We can thus choose $\varepsilon > 0$ arbitrarily small, so that (8) is arbitrarily close to $v_1 F_2(b) - b$ by rightcontinuity of F_2. For $\alpha < 1$ and sufficiently small ε, the payoff from playing $b + \varepsilon$ thus strictly dominates (7). For $\alpha > 0$, an analogous profitable deviation exists for bidder 2. Thus F_1 and F_2 cannot have a simultaneous atom at some $b \in [0, m)$. In particular, $F_1(0) > 0$ implies $F_2(0) = 0$ and vice versa.

To conclude the proof of Step 1, it thus suffices to prove that there does not exist $b \in (0, m)$ with $F_j(b) > F_j(b^-)$ and $F_i(b) = F_i(b^-)$. The proof is again by contradiction and distinguishes two cases. Suppose first that such a b exists and in addition that there exists $\varepsilon > 0$ such that F_i is constant over $(b - \varepsilon, b]$. Since bidder j bids b with positive probability, he must earn his equilibrium payoff from this bid. Yet he can profitably deviate to bidding inside $(b - \varepsilon, b]$ which entails the same probability of winning at lower costs since F_i is constant. Now consider the other case: F_i is non-constant over $(b - \varepsilon, b]$ for any sufficiently small ε. In this case, bidder i earns his equilibrium payoff from bidding slightly below b. This payoff is arbitrarily close to $v_i F_j(b^-) - b$. As in the case of simultaneous atoms, one sees that bidder i can secure himself a payoff arbitrarily close to $v_i F_j(b) - b$ by bidding slightly above b. This is a profitable deviation and thus a contradiction.

Step 2: *There exists $\bar{b} \in [0, m]$ such that F_1 and F_2 are strictly increasing over $[0, \bar{b}]$ and constant over $[\bar{b}, m)$.*

We first observe that if F_i is constant over an interval $(b_1, b_2) \subseteq (0, m)$ then F_j is constant over this interval as well: Suppose otherwise, i.e. there exists an interval over which F_j is increasing while F_i is constant. Then bidder j earns his equilibrium payoff at different points in the interval. These points have identical probability of winning but different bid costs. This leads to a contradiction. Thus the sets over which F_i and F_j are increasing must be identical.

To conclude the proof of Step 2, it suffices to show that if F_i is increasing over an interval (b_1, b_2), F_i is increasing over $(0, b_2)$. Suppose otherwise, i.e., there exist $0 < b_0 < b_1 < b_2$ such that F_1 and F_2 are constant over (b_0, b_1) and increasing over (b_1, b_2). Thus, bidder i earns his equilibrium payoff from bidding b_1. Since F_j is continuous and constant over $[b_0, b_1]$, bidder i has identical probability of winning and smaller bid costs from bidding b_0. This is a profitable deviation.

Step 3: *Consider $\bar{b}$ from Step 2. If $\bar{b} > 0$ then $v_i F_j(b) - b$ is constant in b for $b \in (0, \bar{b})$ and $i \neq j$. In particular, $F'_j(b) = \frac{1}{v_i}$.*

Step 3 follows from the fact that each bidder must earn the same payoff at each point in the interval $(0, \bar{b})$. □

Proof of Corollary 1. (i): By Proposition 1 there are only three candidates for pure equilibria: Both bidders bidding m, or one bidder bidding m while the other bidder bids 0. The latter asymmetric candidates cannot be equilibria since the bidder bidding m can profitably deviate to a smaller bid where he still wins the auction with certainty.

(ii): Consider first an equilibrium in which $c_i > 0$ and $c_j = 0$. In that case bidder i earns an expected equilibrium payoff of zero, since he bids 0 with positive probability while his opponent bids more with probability 1. Similarly, in an equilibrium with $c_i = c_j = 0$ and $\bar{b} > 0$, both bidders must earn their equilibrium payoffs from bids arbitrarily close to 0, implying that both bidders earn 0 in equilibrium.

(iii): If both bidders earn a positive payoff if both bid m, this pair of strategies is an equilibrium since deviating to a smaller bid implies a non-positive expected payoff. Moreover, if bidder i earns a positive expected payoff if j bids m with certainty then bidder i earns a positive expected payoff from bidding m against any bidding strategy of j. This implies uniqueness: If there was another equilibrium, one of the bidders would earn 0 in that equilibrium by (ii). That bidder could profitably deviate to bidding m and earning a strictly positive payoff. □

Proof of Proposition 2. We begin with (iii) where $(\alpha, m) \in C_{III}$ and thus $m < m_1(\alpha)$, i.e.,

$$\alpha_1 v_1 - m > 0 \quad \text{and} \quad \alpha_2 v_2 - m > 0. \tag{9}$$

This implies that each bidder earns a strictly positive expected payoff from bidding m against an opponent who bids m. Thus, both bidders bidding m with certainty is the unique equilibrium by Corollary 1 (iii). The proof of Cases (i) and (ii) proceeds in a number of steps. We first identify a unique equilibrium candidate for each case and then prove that the candidate is indeed an equilibrium. Step 1 shows that any equilibrium must involve mixing over an interval $(0,\bar{b}) \subset [0,m]$:

Step 1: *Let $(\alpha, m) \in C_I \cup C_{II}$. Then any equilibrium is of the form (1) with $\bar{b} > 0$.*

Since $(\alpha, m) \in C_I \cup C_{II}$, one of the conditions in (9) is strictly reversed, i.e., we have $\alpha_1 v_1 < m$ or $\alpha_2 v_2 < m$. Thus, both bidders bidding m is not an equilibrium. It remains to prove that no equilibrium exists in which bidders mix over the set $\{0, m\}$. Suppose otherwise: Since at most one bidder bids 0 with positive probability by Proposition 1, it suffices to consider pairs of strategies where bidder i bids m while bidder j bids 0 with probability $c_j \in (0,1)$ and m otherwise. Thus bidder j earns zero payoff in equilibrium. Since bidder j must be indifferent between bidding 0 and bidding m, it must thus hold that $\alpha_j v_j - m = 0$. Moreover, since one of the conditions in (9) is strictly reversed, we must have $\alpha_i v_i - m < 0$. For sufficiently small $\varepsilon > 0$ and all $c_j \in (0,1)$, bidder i's equilibrium payoff of $(c_j + \alpha_i(1-c_j))v_i - m$ is thus strictly smaller than his payoff of $c_j v_i - \varepsilon$ from bidding ε.

In the remainder of the proof, we first settle the case $\alpha \in (0,1)$. The case $\alpha \in \{0,1\}$ is considered separately in Step 6.

Step 2: *Let $(\alpha, m) \in C_I \cup C_{II}$ with $\alpha \in (0,1)$. If an equilibrium exists, it is given by (1) with parameters satisfying $\bar{b} \in (0,m)$, $d_1 > 0$ and $d_2 > 0$.*

To prove this claim, we have to show that anything else would lead into contradictions. Suppose first that $d_1 = d_2 = 0$ and $\bar{b} \in (0,m]$. The expected equilibrium payoff of bidder i must then equal the payoff from bidding $\bar{b}$ which is given by $v_i - \bar{b} \geq v_i - m > 0$. This contradicts the fact that at least one bidder earns 0 in equilibrium. Thus at least one bidder i must have $d_i > 0$.

Next, suppose that $d_i > 0$ and $\bar{b} = m$. Then, since $\alpha \in (0,1)$, bidder j earns a strictly higher payoff from bidding m than from bids slightly below m. These bids are in the support of his equilibrium strategy since $\bar{b} = m$, implying a contradiction. We thus know that in any equilibrium $d_i > 0$ for at least one bidder and $\bar{b} < m$. It remains to prove that $d_j > 0$. Suppose otherwise, i.e. $d_i > 0$ but $d_j = 0$. Since $\bar{b} < m$, F_j is constant over $[\bar{b}, m]$, so that bidder i can profitably deviate to a bid in $(\bar{b}, m)$ instead of bidding m. We must thus have $d_i > 0$ and $d_j > 0$.

Step 3: *Let $(\alpha, m) \in C_I \cup C_{II}$ with $\alpha \in (0,1)$. Denote by e_I and e_{II} the vectors $(c_1, d_1, c_2, d_2, \bar{b})$ defined, respectively, in (2) and (3). If an equilibrium exists, it is given by (1) with parameters given either by e_I or e_{II}.*

We have seen so far that in equilibrium both bidders mix over $(0,\bar{b})$ with positive probability and bid m with positive probability. Bidders i and j must thus each earn the same equilibrium payoffs π_i and π_j from bidding m, $\bar{b}$ and (slightly above) 0. This yields the conditions

$$\pi_i = (\alpha_i d_j + (1-d_j))v_i - m, \quad \pi_i = (1-d_j)v_i - \bar{b} \quad \text{and} \quad \pi_i = c_j v_i, \tag{10}$$

as well as

$$\pi_j = (\alpha_j d_i + (1-d_i))v_j - m, \quad \pi_j = (1-d_i)v_j - \bar{b} \quad \text{and} \quad \pi_j = c_i v_j. \tag{11}$$

Moreover, we know that in equilibrium at most one bidder bids zero with positive probability, i.e., we either have $c_2 = 0$ or $c_1 = 0$. Each of the two possibilities turns (10) and (11) into a linear system of six equations which can be solved uniquely for the respective remaining six unknowns $\pi_1, \pi_2, c_1, d_1, d_2, \bar{b}$, or $\pi_1, \pi_2, c_2, d_1, d_2, \bar{b}$. The two solutions correspond to the vectors e_I and e_{II}.

The next step establishes a unique equilibrium candidate for each case.

Step 4: *Let $(\alpha, m) \in C_I$ with $\alpha \in (0,1)$. If an equilibrium exists, it is given by (1) with parameters given by e_I. Analogously, let $(\alpha, m) \in C_{II}$ with $\alpha \in (0,1)$. If an equilibrium exists, it is given by (1) with parameters given by e_{II}.*

By Step 3, it suffices to show that e_{II} does not correspond to an equilibrium for $(\alpha, m) \in C_I$ and that e_I does not correspond to an equilibrium for $(\alpha, m) \in C_{II}$. Recall that in equilibrium we must have $\pi_i \geq 0$ and $\pi_j \geq 0$ and that the two equilibrium candidates are derived from the system of equations (10) and (11) under the respective additional constraints $c_j = 0$ and $c_i = 0$. In the following, we determine conditions under which the system of equations (10) and (11) augmented by $c_j = 0$ implies an equilibrium candidate with non-negative values of π_i and π_j. Afterwards, the result follows from symmetry arguments and elementary calculations.

Consider the solution to (10) and (11) augmented by $c_j = 0$. We then have $\pi_i = 0$ and it remains to determine whether $\pi_j \geq 0$. We start with some preliminary calculations: We insert $\pi_i = c_j v_i$ into the first two equations of (10) and solve the resulting two equations for m and $\bar{b}$. We do the same calculation for $\pi_j = c_i v_j$ and (11) and equate the two expressions for m and $\bar{b}$. This gives

$$v_i(1 - c_j - (1-\alpha_i)d_j) = v_j(1 - c_i - (1-\alpha_j)d_i) \quad \text{and} \quad v_i(1 - c_j - d_j) = v_j(1 - c_i - d_i).$$

Subtracting these two equations from each other leads to

$$d_i = d_j \frac{\alpha_i}{\alpha_j} \frac{v_i}{v_j}. \tag{12}$$

From $c_j = 0$ and (10) it follows that

$$d_j = \frac{v_i - m}{v_i(1-\alpha_i)} \quad \text{and} \quad \bar{b} = v_i - \frac{v_i - m}{1-\alpha_i}. \tag{13}$$

We next study under which conditions we have $\pi_j \geq 0$. Using (12), (13) and the fact that $\alpha_i = 1 - \alpha_j$, it follows that

$$\pi_j = v_j(1-d_i) - \bar{b} = v_j - v_i + \left(\frac{1}{\alpha_j} - \frac{\alpha_i}{\alpha_j^2}\right)(v_i - m).$$

Further rewriting the right hand side, we conclude that $\pi_j \geq 0$ is equivalent to

$$\pi_j = v_j - v_i + \frac{1}{\alpha_j}\left(2 - \frac{1}{\alpha_j}\right)(v_i - m) \geq 0.$$

Viewed as a function of $z = 1/\alpha_j$, π_j is a quadratic polynomial with negative leading coefficient. This type of polynomial is non-negative only between its two zeros (if those exist, see below). Solving the quadratic equation we find that $\pi_j \geq 0$ is equivalent to

$$1 - \sqrt{1 + \frac{v_j - v_i}{v_i - m}} \leq \frac{1}{\alpha_j} \leq 1 + \sqrt{1 + \frac{v_j - v_i}{v_i - m}} \tag{14}$$

which is in turn equivalent to

$$-\sqrt{\frac{v_j - m}{v_i - m}} \leq \frac{1 - \alpha_j}{\alpha_j} \leq \sqrt{\frac{v_j - m}{v_i - m}}.$$

The final expression shows that the term under the square-root in (14) is always non-negative. This implies that the quadratic equation indeed possesses two zeros. Thus, the equilibrium candidate we derived under the condition $c_j = 0$ leads to $\pi_i \geq 0$ and $\pi_j \geq 0$ if and only if

$$\frac{\alpha_i}{\alpha_j} \leq \sqrt{\frac{v_j - m}{v_i - m}}. \tag{15}$$

Exchanging the roles of i and j and rearranging terms, we obtain that the equilibrium candidate in the other case $c_i = 0$ leads to $\pi_i \geq 0$ and $\pi_j \geq 0$ if and only if

$$\frac{\alpha_i}{\alpha_j} \geq \sqrt{\frac{v_j - m}{v_i - m}}.$$

Hence, except in the boundary case in which (15) holds with equality, we have identified a unique equilibrium candidate for all $(\alpha, m) \in C_I \cup C_{II}$. It remains to check that the boundary case in which (15) holds with equality is the boundary segment S between C_I and C_{II}. To this end, set $i = 1$, $j = 2$, $\alpha_i = \alpha$ and recall that $v_1 \geq v_2$. Consider the boundary case

$$\frac{\alpha}{1-\alpha} = \sqrt{\frac{v_2 - m}{v_1 - m}}.$$

Since the left hand side is positive, this is equivalent to

$$m\left(1 - \left(\frac{\alpha}{1-\alpha}\right)^2\right) = v_2 - \left(\frac{\alpha}{1-\alpha}\right)^2 v_1, \tag{16}$$

i.e.

$$v_2 = \beta v_1 + (1-\beta)m \quad \text{where} \quad \beta = \left(\frac{\alpha}{1-\alpha}\right)^2.$$

Since $m < v_2 < v_1$, this equation can only hold for $\beta \in (0, 1)$ which is equivalent to $\alpha \in (0, \frac{1}{2})$. For $\alpha < \frac{1}{2}$, (16) can be rewritten to

$$m = v_2 - \frac{\alpha^2}{1 - 2\alpha}(v_1 - v_2), \tag{17}$$

where the right hand side corresponds to m_2 from Definition 1, and thus to the boundary between C_I and C_{II}.

Step 5: *For $(\alpha, m) \in C_I$ with $\alpha \in (0, 1)$ the unique equilibrium is given by (1) with parameters given by e_I. Analogously, for $(\alpha, m) \in C_{II}$ with $\alpha \in (0, 1)$ the unique equilibrium is given by (1) with parameters given by e_{II}.*

By Step 4, it remains to prove that the unique equilibrium candidates are indeed equilibria. In the proof of Step 4 we already saw that the implied equilibrium payoffs are non-negative. By construction, the candidate functions F_i and F_j are increasing over $(0, \bar{b})$ if $\bar{b} \in (0, m)$ and they imply identical payoffs from bidding in $[0, \bar{b})$ and in m. It remains to prove that the candidates are valid distribution functions, i.e., we have $\bar{b} \in (0, m)$, $c_i, d_i \in [0, 1]$ and $c_i + d_i + \bar{b}/v_j = 1$. A straightforward calculation shows that the final normalization condition is fulfilled in all cases. From $\pi_i, \pi_j \geq 0$ it follows that the numbers c_i and d_i are indeed non-negative. It remains to check that $\bar{b} \in (0, m)$. $c_i, d_i \leq 1$ then follows from $\bar{b} \geq 0$ and normalization.

In Case (i), we have $\bar{b} = (m - \alpha v_1)/(1-\alpha)$ so that $\bar{b} < m$ is equivalent to $m < v_1$ and thus satisfied. $\bar{b} > 0$ is equivalent to $m > \alpha v_1$. For $(\alpha, m) \in C_I$ we have $\alpha < \alpha^*$. Thus $m > \alpha v_1$ follows from $m > m_1(\alpha)$. The last assertion holds by the definition of C_I.

In Case (ii), we have $\bar{b} = (m - (1-\alpha)v_2)/\alpha$. $\bar{b} < m$ is thus equivalent to $m < v_2$ and satisfied. $\bar{b} > 0$ is equivalent to $m > (1-\alpha)v_2$. We now distinguish two cases: For $\alpha \geq \alpha^*$ the assertion follows from $m > m_1(\alpha) = (1-\alpha)v_2$ analogously to Case (i). For $(\alpha, m) \in C_{II}$ with $\alpha < \alpha^*$, we have $m > m_2(\alpha)$ by the definition of C_{II}. Observing that $m_2(\alpha) \geq (1-\alpha)v_2$ is equivalent to $\alpha \leq \alpha^*$ concludes the proof of Step 5.

Step 6: *The claim of the proposition also holds for $\alpha \in \{0, 1\}$.*

Suppose $\alpha_i = 0$. Recall that Step 1 of the proof applies so that we have a mixed equilibrium in which one bidder earns zero. Since we have $\alpha_i = 0$ and $m < v_j$, bidder j receives a strictly positive payoff from bidding m regardless of his opponent's strategy, implying $c_j = 0$ and $\pi_i = 0$. Arguing as in the proof of Step 2, we can exclude equilibrium candidates with $d_i = d_j = 0$ and $\bar{b} \in (0, m]$. Likewise, we can exclude candidates where $\bar{b} < m$ and where exactly one bidder has an atom at m, i.e., $d_i > 0$, $d_j = 0$ or $d_j > 0$, $d_i = 0$. We can also rule out candidates with $\bar{b} < m$, $d_i > 0$ and $d_j > 0$ since bidder i would earn more from bidding $\bar{b}$ than from bidding m. Thus $\bar{b} = m$.

Solving the conditions $\bar{b} = m$, $c_j = 0$ and the normalization condition $c_j + d_j + \bar{b}/v_i$ for d_j pins down a unique candidate for the equilibrium strategy of bidder j. To determine the strategy of bidder i, observe that bidder j's equilibrium payoff must equal his payoff from bidding m which is given by $\pi_j = v_j - m$. Arguing as in Step 3 yields $\pi_j = c_i v_j$, and thus $c_i = 1 - m/v_j$. The condition $c_i + d_i + \bar{b}/v_j = 1$ implies $d_i = 0$ since $c_i + m/v_j = 1$. This identifies a unique equilibrium candidate. It is straightforward to see that it is indeed an equilibrium and that it corresponds to the special cases $\alpha = 0$ of Case (i) and $\alpha = 1$ of Case (ii). □

Proof of Proposition 3. (i): Recall that $(\alpha, m) \in S$ is the boundary case between Cases (i) and (ii) of Proposition 2. The proof of that proposition implicitly contains the result that the two equilibrium candidates given in (2) and (3) are equilibria for $(\alpha, m) \in S$. Uniqueness follows from checking that the two equilibria coincide for $(\alpha, m) \in S$ and correspond to the candidate given in (i).

(ii) and (iv): We have $\alpha_i v_i = m$ and $\alpha_j v_j > m$ with $(i, j) = (1, 2)$ in Case (ii) and $(i, j) = (2, 1)$ in Case (iv). Thus, both bidders bidding m is a pure equilibrium in which only bidder j makes a strictly positive payoff. Hence there does not exist an equilibrium in which both bidders earn a strictly positive payoff by Corollary 1. Since bidder j can secure a strictly positive payoff against any strategy of i, we must have $c_j = 0$, $\pi_j > 0$ and $\pi_i = 0$. $\alpha_i v_i = m$ implies that bidder i can earn a strictly positive payoff from bidding m unless $d_j = 1$. Thus $d_j = 1$ and $\bar{b} = 0$ in any equilibrium. To identify all equilibria, it suffices to identify the combinations of c_i and $d_i = 1 - c_i$ which correspond to equilibria. Bidder i earns zero from bidding 0 and from bidding m and a negative payoff from bidding in between. He thus cannot profitably deviate. Bidder j's most profitable deviation is always given by bidding slightly above 0, thereby winning the auction with probability c_i at arbitrarily small costs. $c_i \in [0, 1]$ yields an equilibrium whenever bidding m dominates this deviation. This holds if and only if

$$c_i v_j \leq (c_i + \alpha_j(1 - c_i))v_j - m, \tag{18}$$

which is equivalent to $c_i \leq 1 - m/(\alpha_j v_j)$. This concludes the proof.

(iii): The proof is similar to that of (ii) and (iv). We have $\alpha_i v_i = m$ and $\alpha_j v_j = m$, implying that $d_i = d_j = 1$ yields an equilibrium in which both bidders earn zero payoff. There thus does not exist an equilibrium where both bidders earn a positive payoff. In any equilibrium with $d_i < 1$ and $d_j < 1$, both bidders would earn a strictly positive payoff from bidding m since $\alpha_i v_i = m$ and $\alpha_j v_j = m$. Thus $\bar{b} = 0$ and $d_i = 1$ or $d_j = 1$ holds in any equilibrium. We studied the remaining equilibrium candidates in the proof of (ii) and (iv): To rule them out, it suffices to verify that condition (18) and the symmetric condition with i and j interchanged collapse to $c_i \leq 0$ and $c_j \leq 0$ in Case (iii).

(v) and (vi): Arguing as in the proof of Step 2 of Proposition 2, we see that there do not exist equilibria with $\bar{b} < m$ and $d_1 = d_2 = 0$ or $d_1 > 0, d_2 = 0$ or $d_2 > 0, d_1 = 0$. Furthermore, $\bar{b} < m$ cannot hold in equilibrium: Consider the remaining case $d_1 > 0$, $d_2 > 0$ and $\bar{b} < m$. If $\alpha = 0$, bidder 1 can profitably deviate from bidding m to bidding $\bar{b}$. If $\alpha > 0$, bidder 2 earns a strictly negative payoff from bidding m since $v_2 = m$, a contradiction to equilibrium. We thus have $\bar{b} = m = v_2$. Bidder 1's normalization condition $c_1 + d_1 + \bar{b}/v_2 = 1$ yields $c_1 = d_1 = 0$ in any equilibrium. This uniquely pins down his equilibrium strategy. For bidder 2, normalization implies $c_2 + d_2 = 1 - v_2/v_1$. Hence, there cannot exist any equilibria in addition to the candidates listed in Case (vi). In Case (vi), a straightforward calculation shows that all the candidates indeed correspond to equilibria. In Case (v), we can further rule out $d_2 > 0$: Since $\bar{b} = m$, bidder 1 earns his equilibrium payoff from bids right below m. Yet if $d_2 > 0$ and $\alpha > 0$, bidder 1 can profitably deviate to bidding m. □

Proof of Lemma 1. Using the explicit expressions for c_i, d_i and $\bar{b}$ given in Cases (i) and (ii) of Proposition 2 it follows for $(\alpha, m) \in C_I$ that

$$\frac{d\sigma(\alpha, m)}{dm} = \frac{(1-2\alpha)(m(v_1 - v_2) + v_1 v_2)}{(1-\alpha)^2 v_1 v_2}.$$

This expression is positive since $\alpha < \frac{1}{2}$ for $(\alpha, m) \in C_I$ and $v_1 > v_2$. For $(\alpha, m) \in C_{II}$ we obtain

$$\frac{d\sigma(\alpha,m)}{dm} = \frac{(2\alpha-1)(v_1(v_2-m)+mv_2)}{\alpha^2 v_1 v_2}.$$

Since $v_2 > m$, this derivative is positive for $\alpha > \frac{1}{2}$ and negative for $\alpha < \frac{1}{2}$. $\square$

Proof of Proposition 4. We first prove the following claim: If (α, m) is a solution to the optimization problem, then $(\alpha, m) \in \bar{S} = S \cup \{(0, v_2)\} \cup \{(\alpha^*, m^*)\}$.

Recall that (α^*, m^*) is strictly optimal for the designer within $T \cup C_{III}$. Furthermore, (α^*, m^*) is strictly better than the unrestricted all-pay auction and thus $\sigma(\alpha^*, m^*) > \sigma(\alpha, m)$ for all $(\alpha, m) \in U$ with $\alpha > 0$. Observe that the equilibrium non-uniqueness at $(\alpha, m) \in T \setminus (\alpha^*, m^*)$ is such that probability mass is shifted from m to zero as (α, m) moves from C_{III} to $C_I \cup C_{II}$ for fixed α and increasing m. Thus, the function $\sigma(\alpha, m)$ makes a downwards jump when passing the line T for fixed α and increasing m (except in the case $\alpha = \alpha^*$).

Now keep α fixed and increase m from 0 to v_2. We distinguish three cases: $\alpha > \frac{1}{2}$, $\alpha \in [\alpha^*, \frac{1}{2}]$ and $\alpha < \alpha^*$. Consider first the case $\alpha > \frac{1}{2}$. By Lemma 1, $\sigma(\alpha, m)$ is increasing until T is reached. There it jumps downwards and then increases again until $m = v_2$. This yields two local maxima and thus two potential candidates for optimal values of $\sigma(\alpha, \cdot)$: The unique (α, m) with $(\alpha, m) \in T$ and $(\alpha, v_2) \in U$. Both points are dominated by (α^*, m^*).

Now consider a fixed $\alpha \in (\alpha^*, \frac{1}{2}]$. In this case $\sigma(\alpha, m)$ is increasing in m until T is reached. Then there is a downwards jump and then $\sigma(\alpha, m)$ decreases until U is reached. In this case, there is a unique maximizer m of $\sigma(\alpha, \cdot)$ with $(\alpha, m) \in T$. Again, (α^*, m^*) dominates.

Finally consider the case of $\alpha \leq \alpha^*$. Similar to the first case, there are two local maximizers of $\sigma(\alpha, \cdot)$, one with $(\alpha, m) \in T$ and one with $(\alpha, m) \in S$. For the first one it is clear that it is dominated by (α^*, m^*). Thus global maximizers must lie in $\bar{S}$ and the prove of the claim is complete.

It remains to maximize $\sigma(\alpha, m_2(\alpha))$ in $\alpha \in [0, \alpha^*]$ since $\bar{S}$ can be written as $\bar{S} = \{(\alpha, m) | \alpha \in [0, \alpha^*], m = m_2(\alpha)\}$. We calculate

$$\frac{d\sigma(\alpha, m_2(\alpha))}{d\alpha} = \frac{\alpha(1-\alpha)(v_1-v_2)^3}{(1-2\alpha)v_1 v_2} > 0.$$

Thus it is optimal to choose α as large as possible: $(\alpha^*, m_2(\alpha^*)) = (\alpha^*, m^*)$ is a global optimizer. $\square$

Proof of Proposition 5. For $\alpha = \frac{1}{2}$ the optimal value $m = \frac{v_2}{2}$ is known from Che and Gale (1998). For $\alpha \in \{0, 1\}$ it suffices to consider $m = v_2$ by Lemma 1. A direct comparison of the three candidates yields the result of the proposition. $\square$

Proof of Proposition 6. The proofs of Proposition 1, of Corollary 1, of Steps 1, 2, 3, and 6 of Proposition 2 and of Proposition 3 (ii–iv) do not rely on the asymmetry assumption and still apply in the symmetric case. This proves part (iii) of Proposition 6. It also implies that the strategy pairs identified in Step 3 of the proof of Proposition 2 are the only equilibrium candidates for $(\alpha, m) \in C \setminus (C_{III} \cup T)$ with $m < v$. These two candidates are stated in parts (i) and (ii) of Proposition 6 for the symmetric case. The equilibrium payoffs π_i stated in part (i) are non-negative for both bidders if and only if $\alpha \leq \frac{1}{2}$. The equilibrium payoffs π_i stated in part (ii) are non-negative for both bidders if and only if $\alpha \geq \frac{1}{2}$. Since the two equilibrium candidates coincide for $\alpha = \frac{1}{2}$, it follows that we have identified a unique equilibrium candidate for both $\alpha \leq \frac{1}{2}$ and $\alpha > \frac{1}{2}$. The proof that the two candidates define valid distribution functions is similar to the proof of Step 5 of Proposition 2. It remains to consider the case $v = m$. It follows like in the proof of Proposition 3 (v) that $\bar{b} = v$ in this case. The normalization conditions $c_i + d_i + \bar{b}/v = 1$ thus imply $c_i = d_i = 0$ for both bidders. We have thus identified the equilibrium of the unconstrained all-pay auction as the unique equilibrium candidate. $\square$

References

Baye, Michael, Kovenock, Dan, de Vries, Casper, 1996. The all-pay-auction with complete information. Econ. Theory 8 (2), 91–305.
Che, Yeon-Koo, Gale, Ian, 1998. Caps on political lobbying. Amer. Econ. Rev. 88 (3), 643–651.
Gavious, Arieh, Moldovanu, Benny, Sela, Aner, 2003. Bid costs and endogenous bid caps. RAND J. Econ. 33 (4), 709–722.
Hillman, Arye L., Riley, John G., 1989. Politically contestable rents and transfers. Econ. Politics 1 (1), 17–39.
Hillman, Arye L., Samet, Dov, 1987. Dissipation of contestable rents by small numbers of contenders. Public Choice 54 (1), 63–82.
Hart, Sergiu, 2014. Allocation games with caps: from Captain Lotto to all-pay auctions. Discussion Paper 670. Center for Rationality, Hebrew University.
Konrad, Kai A., 2009. Strategy and Dynamics in Contests. Oxford University Press, Oxford.
Persico, Nicola, Sahuguet, Nicolas, 2006. Campaign spending regulation in a model of redistributive politics. Econ. Theory 28 (1), 95–124.
Sahuguet, Nicolas, 2006. Caps in asymmetric all-pay auctions with incomplete information. Econ. Bull. 3 (9), 1–8.
Siegel, Ron, 2009. All-pay contests. Econometrica 77 (1), 71–92.
Weber, Robert J., 1985. Auctions and competitive bidding. In: Peyton Young, H. (Ed.), Fair Allocation. American Mathematical Society, Providence RI, pp. 143–170.