



Rebates in a Bertrand game

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ABSTRACT

We study a price competition game in which customers are heterogeneous in the rebates they get from either of two firms. We characterize the transition between competitive pricing (without rebates), mixed strategy equilibrium (for intermediate rebates), and monopoly pricing (for larger rebates).

In the mixed equilibrium, a firm's support consists of two parts: (i) aggressive prices that can steal away customers from the other firm, and (ii) defensive prices that can only attract customers who get the rebate. Both firms earn positive expected profits.

We show that, counter-intuitively, for intermediate rebates, an increase in rebates leads to a lower market segmentation.

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1. Introduction

The presumably simplest – and in this sense most fundamental – model on rebates has not yet been fully analyzed. Klemperer (1987a, Section 2), studies a situation in which two firms with equal and constant marginal costs compete in prices. He frames the example as one of the airline industry, where rebates are given. Each customer has to pay the full price at one firm if he/she buys there, but only the reduced price if he/she buys from the other firm. Klemperer shows that, for certain parameter constellations, there is an equilibrium in pure strategies where each customer buys from the firm where he/she can get the rebate and the reduced price equals the monopoly price. Therefore, firms earn monopoly profits in their segments.

The reason why the model has not been further analyzed may be that, unless the rebates are sufficiently high, an equilibrium in pure strategies fails to exist. Therefore, the literature has attached further components to the model to guarantee the existence of pure strategy equilibria.^{2,3} We analyze the “innocent” model

without any restriction on the size of the rebates. We show that, when customers differ in the rebates they can get, both firms earn positive expected profits.

Studying the effects of rebates is highly important, because rebates are ubiquitous for customers: for example, most airlines offer rebates in the form of miles via frequent-flyer programs; in 2011, U.S. customers used coupons to save 4.6 billion dollars on consumer package goods.⁴

In the main part of our analysis, we focus on unit demand. The equilibrium is characterized by three different regimes. First, when the rebates are small, the Nash equilibrium is in mixed strategies without mass points. Second, for intermediate levels of rebate, the equilibrium is still in mixed strategies, but there is a mass point at the upper end of the support. Third, when the rebates are high, the equilibrium is in pure strategies, just as in Klemperer (1987a). In the first two regimes, firms mix between two types of strategy: an aggressive one and a defensive one. Either a firm charges low prices, which attracts all customers of its home base for sure, and with some probability the other customers as well. Or a firm charges high prices, thus risking losing the customers of its home base, but earning a high payoff if it still attracts them. For the case where firms mix without atoms we show that the probabilities of attacking and defending stand in the celebrated golden ratio.

Furthermore, we study market segmentation, i.e., the probability that a customer buys at the firm where he/she gets the rebate.

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² Klemperer (1995, footnote 7): “Pure-strategy equilibrium can be restored either by incorporating some real (functional) differentiation between products (Klemperer (1987b)), or by modeling switching costs as continuously distributed on a range including zero (...) (Klemperer (1987a))”. Banerjee and Summers (1987) consider a sequential price setting to circumvent mixed strategies. Also, Caminal and Matutes (1990) analyze a setting with real differentiation.

³ Mixed strategy equilibria often arise in oligopoly pricing models. For example, in Padilla's (1992) dynamic setting with myopic customers; in Deneckere et al.

(1992), who analyze a game with loyal customers and without rebates; in Beckmann's (1965) and Allen and Hellwig's (1986, 1989, 1993) Bertrand–Edgeworth models, where capacity-constrained firms choose prices.

⁴ See <https://www.nchmarketing.com/ResourceCenter/default.aspx>.

We show that – counter-intuitively at first sight – market segmentation may decrease in rebates. This happens when the rebates reach an intermediate level at which the customers' limited willingness to pay starts to affect the firms' pricing behavior. From this level of rebate on, firms have to concentrate some mass of their pricing strategy into an atom at the upper end of their price interval. At that price, the firm can only attract its home base if the other firm does not attack. Because these defensive strategies have the effect that customers buy from the firm where they cannot get a rebate whenever this firm offers an aggressive price, the segmentation of the market is decreasing in the level of the rebates. When the rebates get large, however, firms play aggressive prices with a diminishing probability. Then the market segmentation increases again and finally converges to full segmentation.

We also study the normative aspects of our model. We show that rebates deteriorate customer and total welfare. We also demonstrate that customers face a coordination problem: they are collectively worse off when there are rebate systems, but individually they are better off when they participate in a system than when they do not.

There is a large literature which shows that firms can yield positive profits also if they compete in prices: in the classical paper by Hotelling (1929), customers are located at different places, which can also be interpreted that customers differ in their tastes (i.e., there is horizontal product differentiation); Shaked and Sutton (1982) show that firms can relax price competition via quality differentiation (i.e., there is vertical product differentiation). In our paper, all customers evaluate the quality of the firms' products equally. Yet one can interpret customers who can get rebates to have preferences for some firm. However, in our model, firms care which customers buy from them because some customers have to pay the full price, while others can get a rebate and thus pay a lower price. This is in contrast to product differentiation models, where firms do not care to whom they sell. Thus, our model cannot be interpreted as a product differentiation model.

Bester and Petrakis (1996) study the effects of coupons/rebates on price setting in a one-period model where firms can target certain customers. In equilibrium, each firm sends coupons to customers who live in the "other city". Therefore, unlike in our model, coupons reduce the firms' profits. For similar models, see Shaffer and Zhang (2000) and Chen (1997).

Technically, we also contribute to the literature studying mixed equilibria of asymmetric auction-type games; see Siegel (2009, 2010). Unlike in the models studied for example by Siegel, in our setting none of the boundaries of the pricing interval can easily be inferred a priori. Instead, we determine the equilibrium by imposing conditions on the relation between upper and lower boundaries. This way, we can explicitly determine equilibria of a natural class of asymmetric auctions. Interpreted as an auction, our model is a complete-information first-price (procurement) auction in which bidders are asymmetric regarding their stochastic bidding advantages.

The paper proceeds as follows. In Section 2, we introduce the model. In Section 3, we solve the equilibrium explicitly for the case of unit demand. In Section 4, we characterize the equilibrium for a large variety of demand functions. In Section 5, we discuss endogenous rebates and provide a simulation study. In Section 6, we offer a concluding discussion. The proofs are relegated to the Appendix.

2. The model

We analyze a market with two firms and a continuum of customers. The customers are of one of two types: a mass m_1 of customers gets a rebate $r_1 \geq 0$ at firm 1 and no rebate at firm 2. We call this group of customers the "home base" of firm 1. A mass

m_2 of customers gets no rebate at firm 1 and a fixed rebate $r_2 \geq 0$ at firm 2.⁵ Each customer wants to buy exactly one object, for which his/her valuation is \bar{p} . Both firms produce these objects at the same unit costs, which are normalized to zero. Firms engage in price competition, i.e., simultaneously set gross prices. Then customers buy from the firm where they have to pay the lower net price (i.e., price minus rebate), provided that this net price is below their valuation. In Section 4, we will extend our analysis to much more general demand functions and to situations where not all customers get rebates.

Let us start with an intuition why in this game the Bertrand Paradox does not arise, i.e., why firms must earn positive profits. When a firm offers a rebate, it has to charge gross prices well above zero to obtain no loss. This enables the other firm to earn a positive profit. Hence, in equilibrium, the other firm also charges prices well above zero, which in turn allows the former firm to earn a positive profit, too.

Klemperer (1987a) obtains essentially the following partial result.

Proposition 1. *Suppose that $m_1 > 0$ and $m_2 > 0$. Then, if r_1 and r_2 are sufficiently large, each firm earns monopoly profits in its market segment.*

In the following sections, we explore what happens if the rebates are not that high, such that the above pure strategy equilibrium does not exist.

3. Characterization of equilibria

In the following, we show that, if the rebates are moderate, a mixed strategy equilibrium arises. Section 3.1 characterizes the mixed strategy equilibrium for the case where the rebates are small enough to ensure that \bar{p} does not interfere with the firms' pricing strategies: in this case, each firm i mixes over strictly positive prices that are strictly lower than $\bar{p} + r_i$. Section 3.2 gives a complete characterization of the transition between the pure and the mixed strategy equilibrium for the symmetric case $r_i = r_j$ and $m_i = m_j$. Section 3.3 introduces customers who cannot get rebates at any firm, and shows that these make the firms' competition behavior much harsher.

3.1. Atomless pricing for moderate rebates

Denote by F_i the distribution function underlying the mixed price-setting strategy of firm i , and let π_i be firm i 's equilibrium payoff. Then in equilibrium it has to hold that, for all $p \in \text{supp}F_i$,

$$\pi_i = m_i(p - r_i)(1 - F_j(p - r_i)) + m_j p(1 - F_j(p + r_j)). \quad (1)$$

The equilibrium distributions we identify are characterized as follows: firms mix between two types of strategy – an aggressive one and a defensive one. Either a firm charges low prices, attracts all customers of its home base for sure, and with some probability attracts the other customers as well. Or a firm charges high prices, thus running the risk of losing the customers of its home base, but earning a high payoff if it still retains them. Formally, F_i can be written as $q_i A_i + (1 - q_i) D_i$, where A_i and D_i are distribution functions and $q_i \in [0, 1]$. We call $q_i \in [0, 1]$ the "attack probability", as only a firm playing the aggressive strategy may attract customers of the other firm's home base: A_i (the aggressive strategy) and D_i (the defensive strategy) have distinct supports $[a_i, \bar{a}_i]$ and $[d_i, \bar{d}_i]$ with $\bar{a}_i \leq \bar{d}_i$.

⁵ In the paper, we mostly treat rebates as given. In Section 5, we discuss endogenous rebates.

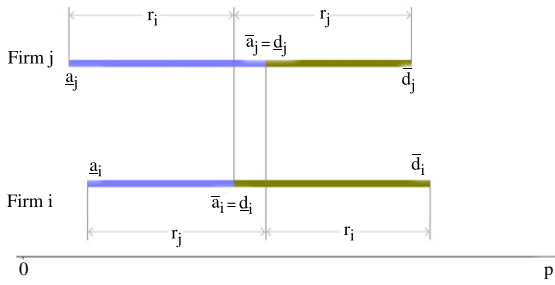


Fig. 1. The boundaries of the strategy supports.

Fig. 1 schematically depicts the supports of the two firms' strategies in an example with $r_i > r_j$. Given this decomposition of the firms' strategies, (1) becomes, for small p , that is, for $p \in [a_i, \bar{a}_i]$,

$$\pi_i = m_i(p - r_i) + m_j p(1 - q_j)(1 - D_j(p + r_j)), \tag{2}$$

and for larger p , that is, for $p \in [\bar{d}_i, \bar{d}_i]$,

$$\pi_i = m_i(p - r_i)(1 - q_j A_j(p - r_i)). \tag{3}$$

Our first main result provides an explicit characterization for an equilibrium under the assumption that the maximal willingness to pay, \bar{p} , is sufficiently large not to interfere with the firms' pricing strategies.

Proposition 2. Assume that \bar{p} is sufficiently large (i.e., $\bar{p} > \max\{\bar{d}_1 + r_1, \bar{d}_2 + r_2\}$, where \bar{d}_j is defined below). Then an equilibrium is given as follows: equilibrium attack probabilities q_j and equilibrium payoffs π_j are

$$q_j = \frac{m_i^2 + m_i m_j + m_j^2 - \psi(m_i, m_j)(m_i^2 - m_i m_j + m_j^2)}{2m_j^2} \tag{4}$$

and

$$\pi_j = \frac{(\psi(m_i, m_j) + 1)m_i m_j - (\psi(m_i, m_j) - 1)m_j^2}{2m_i} r_i + \frac{(\psi(m_i, m_j) - 1)m_j}{2} r_j,$$

where

$$\psi(m_i, m_j) = \sqrt{\frac{m_i^2 + 3m_i m_j + m_j^2}{m_i^2 - m_i m_j + m_j^2}}.$$

The equilibrium strategies consist of the defensive strategy

$$D_j(p) = 1 - \frac{\pi_i - m_i(p - r_i - r_j)}{m_j(p - r_j)(1 - q_j)} \tag{5}$$

and the aggressive strategy

$$A_j(p) = \frac{1}{q_j} \left(1 - \frac{\pi_i}{m_i p} \right), \tag{6}$$

with supports given by

$$\underline{d}_j = \frac{\pi_i + m_i(r_i + r_j) + r_j m_j(1 - q_j)}{m_i + m_j(1 - q_j)}, \quad \bar{d}_j = \frac{\pi_i}{m_i} + r_i + r_j$$

and

$$\underline{a}_j = \frac{\pi_i}{m_i}, \quad \bar{a}_j = \frac{\pi_i}{m_i(1 - q_j)}.$$

Furthermore, the supports of the equilibrium strategies are connected, i.e., $\underline{d}_j = \bar{a}_j$. The aggressive strategy of firm i is a downward shift by r_j of the defensive strategy of firm j , i.e., $D_j(p + r_j) = A_i(p)$.

Our basic approach towards identifying these equilibria is to postulate that the equilibrium strategies have identical supports up to correcting for shifts due to the rebates. While we do not address the uniqueness of the equilibrium in this paper, let us note that this approach is in line with a well-known finding about simpler competition games with price dispersion such as Varian's (1980) model of sales. In all equilibria, firms do not mix over intervals where all opponents are inactive. In the two-player case, this necessary equilibrium condition is restrictive enough to rule out multiplicities of equilibria in all similar games we are aware of.

The fact that the aggressive strategy of player i is identical, up to a shift by r_j , to the defensive strategy of player j , has the following consequence: given that firm i attacks and firm j defends, there is a probability of $1/2$ that all customers end up at firm i . With the complementary probability, all customers buy at their home firm.

While the dependence of the equilibrium on the group sizes m_i and m_j is a bit more complex, the dependence on the rebates is very simple: the attack probabilities q_j are independent of the rebates. The equilibrium payoffs are linearly increasing in both rebates. The function ψ which determines equilibrium payoffs and attack probabilities is a symmetric function which only depends on the ratio of m_i and m_j . It takes its maximum value of $\sqrt{5}$ for $m_i = m_j$ and decreases to the value 1 as m_i/m_j goes to 0 or ∞ .

To see how asymmetries in the attack probabilities are linked to asymmetries in group sizes, observe from (4) that the following relation holds:

$$q_i m_i^2 = q_j m_j^2.$$

Intuitively, a firm that gives rebates only to a few customers is more inclined to set small prices, targeting customers who get a rebate from the other firm.

To illustrate the proposition, consider the case $m_i = m_j = 1$. Then, the equilibrium is given by

$$q_i = q = \frac{3 - \sqrt{5}}{2} \approx 0.382 \quad \text{and} \quad \pi_i = r_j + (1 - q)r_i.$$

Note that this implies that the probabilities of attacking and defending stand in the celebrated golden ratio, i.e.,

$$\frac{1 - q}{q} = \frac{1 + \sqrt{5}}{2}.$$

To get some intuition for the equilibrium – and also for the occurrence of the golden ratio – let us consider the special case $r_i = r_j = r$. Let us assume that in equilibrium both players mix with some atomless strategy over an interval of length $2r$, i.e., $[\underline{a}, \underline{a} + 2r]$. Let q be the equilibrium attack probability, i.e., the probability mass in the lower half $[\underline{a}, \underline{a} + r]$.

We demonstrate now how these assumptions uniquely determine the equilibrium values of \underline{a} and q and the equilibrium payoffs. Let us compare the firms' expected payoffs from playing prices \underline{a} , $\underline{a} + r$, and $\underline{a} + 2r$, which in equilibrium must be identical. Note first that, by playing a price of $\underline{a} + r$, a firm attracts all customers from its home base, but no customers from the opponent's home base. Thus

$$\pi(\underline{a} + r) = \underline{a} + r - r = \underline{a}.$$

Compare to this playing a price of \underline{a} . Then our firm still attracts its home base with certainty but payments from the home base decrease by r . Yet unlike before, our firm receives \underline{a} from the customers in the other firm's home base as well, provided that the other firm plays a price above $\underline{a} + r$, which happens with probability $1 - q$. Thus, from $\pi(\underline{a} + r) = \pi(\underline{a})$, we can conclude that the advantages and disadvantages from switching from $\underline{a} + r$ to \underline{a} must cancel out in equilibrium, i.e.,

$$r = (1 - q)\underline{a}. \tag{7}$$

Now, consider the payoff from playing a price of $\underline{a} + 2r$. In this case, our firm attracts its home base only if the other firm plays a price above $\underline{a} + r$, which happens with probability $1 - q$. We hence get $\pi(\underline{a} + 2r) = (1 - q)(\underline{a} + 2r - r) = (1 - q)(\underline{a} + r)$.

As $\pi(\underline{a} + 2r)$ and $\pi(\underline{a} + r)$ must be identical in equilibrium, we get $\underline{a} = (1 - q)(\underline{a} + r)$. (8)

Now, let us compare (7) and (8). From these two equations, we see that the ratio between r and \underline{a} is the same as the ratio between \underline{a} and $\underline{a} + r$. This is exactly the defining property of the golden ratio, implying that

$$\frac{\underline{a}}{r} = \frac{1 + \sqrt{5}}{2},$$

and thus, by (7),

$$q = \frac{3 - \sqrt{5}}{2}.$$

3.2. From Bertrand to monopoly

So far, we have analyzed the cases of sufficiently large and of sufficiently small rebates, giving rise to, respectively, a pure strategy equilibrium in $\bar{p} + r$ or a mixed strategy equilibrium. For the symmetric case, we now round out the analysis by characterizing the equilibrium also for intermediate values of r . This equilibrium is composed of an atom in $\bar{p} + r$ and mixing below this price. A gap arises between the supports of the aggressive and the defensive strategies. The transition between the different types of equilibrium is continuous in r .

Proposition 3. Assume that $m_i = m_j = 1$, $\bar{p} = 1$, and $r_1 = r_2 = r$.

- (i) For $r \leq r^* := \frac{3-\sqrt{5}}{2}$, Proposition 2 characterizes an equilibrium with $q = \frac{3-\sqrt{5}}{2}$ and $\pi = (2 - q)r$.
- (ii) If $r^* \leq r \leq 1$, an equilibrium is given as follows: both firms play the aggressive strategy $A(p)$ with probability q^A , the defensive strategy $D(p)$ with probability q^D , and a price of $1 + r$ with the remaining probability. The probabilities q^A and q^D and the equilibrium payoffs π are given by

$$q^A = 1 - \sqrt{r}, \quad q^D = 1 - r \quad \text{and} \quad \pi = \sqrt{r}.$$

The distribution functions A and D are given by

$$A(p) = \frac{1}{q^A} \left(1 - \frac{1 - q^A}{p} \right) \quad \text{and}$$

$$D(p) = \frac{1}{q^D} \left(1 - q^A - \frac{1 - q^A - p + 2r}{p - r} \right).$$

The supports of A and D are defined through

$$\underline{a}_j = \sqrt{r}, \quad \bar{a}_j = 1,$$

and

$$\underline{d}_j = \sqrt{r} + r, \quad \bar{d}_j = 1 + r.$$

- (iii) If $r \geq 1$, a pure strategy equilibrium arises where both firms set a price of $1 + r$. Each firm earns an equilibrium payoff of 1.

It is straightforward to generalize Proposition 3 to $m_i = m_j \neq 1$ and $\bar{p} \neq 1$. Furthermore, it is easy to verify that Cases (i) and (ii) coincide for $r = \frac{3-\sqrt{5}}{2}$. Likewise, for $r = 1$, the equilibrium of Case (ii) degenerates to an atom in $1 + r = 2$.

Figs. 2 and 3 illustrate Proposition 3. The upper quadrangle in Fig. 2 shows the dependence of the support of the firms' defensive

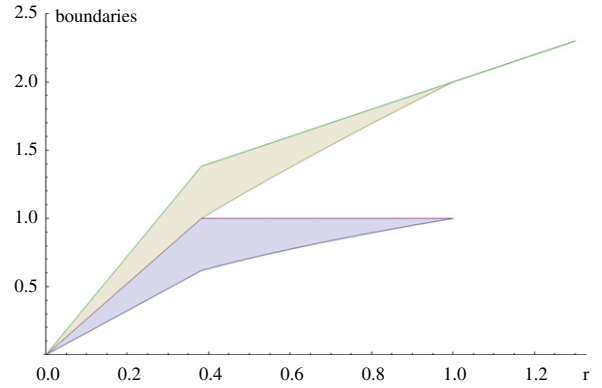


Fig. 2. The strategy supports $\bar{d} \geq d \geq \bar{a} \geq a$ as functions of r .

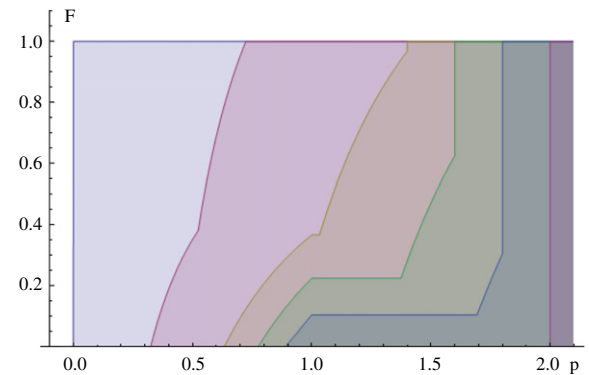


Fig. 3. The pricing strategy $F(p)$ for $r = 0, 0.2, 0.4, \dots, 1$.

strategy on r . The upper bound corresponds to \bar{d} , and the lower bound to \underline{d} . The lower quadrangle depicts the support of the aggressive strategy, where the upper and lower bound correspond to \bar{a} and \underline{a} , respectively. Up to $r^* \approx 0.382$, the curves are the same as in the case of unrestricted willingness to pay. Yet once the curve \bar{d} reaches the value $1 + r^*$, the limited willingness to pay of the customers gets important: from there on, \bar{d} increases less, and stays always equal to $1 + r$, the maximal willingness to pay of the home-base customers. Firms put an atom on \bar{d} from the kink onwards. The distance between \bar{a} and \bar{d} is always r , as is the distance between \underline{a} and \underline{d} . That is, r is the maximal markup a firm can charge from its home base. The pricing strategies converge to the case of a segmented market with monopolistic prices as r approaches 1.

Fig. 3 shows the distribution functions of the firms' pricing strategies for different values of r ($r = 0, 0.2, 0.4, \dots, 1$). We see the interpolation between competitive pricing ($r = 0$), where firms set prices of 0, and full segmentation (for $r = 1$), where both firms set a price of $1 + r = 2$ with certainty. For $r > r^*$, the pricing strategies have a gap between the aggressive and the defensive prices, corresponding to the constant part in the distribution functions. The mass of the atom corresponds to the size of the jump in the distribution functions. For $r = 0.2 < r^*$, the kink in the curve marks the boundary between aggressive and defensive pricing.

The firms' profits increase linearly in r for r low and sublinearly for intermediate r . When $r \geq 1$, the profits stay constant in r . Intuitively, once the market is fully segmented, firms cannot earn more than monopoly profits; hence they do not gain from higher rebates.

Fig. 4 shows the segmentation probability, i.e., the probability that all customers buy where they get the rebate, as a function of r . Note first that even arbitrarily small rebates are sufficient

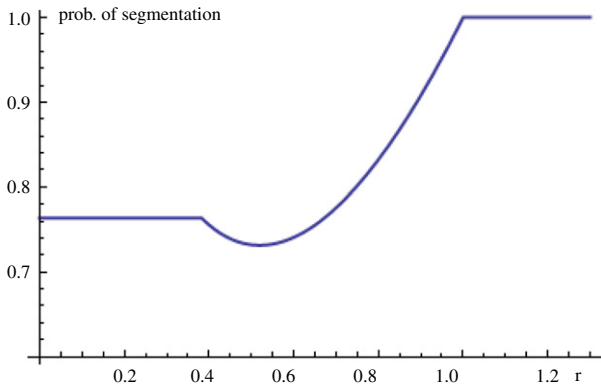


Fig. 4. The probability of market segmentation as a function of r .

to generate a high segmentation probability. Interestingly, the probability that the market is segmented is not monotonically increasing in r . Rather, the segmentation probability is constant until $r = r^*$, then decreases for some interval until it increases again, reaching the value 1 for $r \geq 1$. To get an intuition for this behavior, note first that the probability of no segmentation is the same as the probability of a successful attack. Now, in Cases (i) and (ii) of Proposition 3, we can argue as in the proof of Proposition 2 that $A(p) = D(p+r)$. Therefore, given that one firm attacks and the other defends, the probability of a successful attack is 1/2. Observe also that playing an atom in \bar{d} can be interpreted as deciding not to defend but to rely on the cases where the opponent does not attack. We thus get the following: for $r < r^*$, the segmentation probability is independent of r , as it only depends on q , which is independent of r . For $r \geq r^*$, the firms set an atom in \bar{d} , which implies that the probability of success of an attack increases. This effect drives the segmentation probability down. Yet as r further approaches 1, the fact that attacks become increasingly rare takes over, and the segmentation probability approaches 1.

3.3. Customers without rebates

We now introduce a mass $m_0 > 0$ of customers who do not receive a rebate from any firm. While it is generally difficult to find explicit equilibria for this case, we can provide a solution for a symmetric case with sufficiently many m_0 -customers. This leads to a number of interesting conclusions and comparisons. Let $m_1 = m_2 = m_h > 0, m_0 > 0, r_1 = r_2 = r$. Assume that customers have an infinite (or sufficiently large) willingness to pay. Then we find the following equilibrium.

Proposition 4. *If*

$$m_0 \geq \frac{m_h}{\alpha},$$

where

$$\alpha = \frac{1}{6} \left(2 + 2^{\frac{2}{3}} (47 - 3\sqrt{93})^{\frac{1}{3}} + 2^{\frac{2}{3}} (47 + 3\sqrt{93})^{\frac{1}{3}} \right) \approx 2.15,$$

then a symmetric equilibrium is given by both firms mixing over $S = [\frac{m_h}{m_0}r, (1 + \frac{m_h}{m_0})r]$, with distribution function

$$F(p) = \left(1 + \frac{m_h}{m_0} \right) - \frac{rm_h \left(1 + \frac{m_h}{m_0} \right)}{m_0 p}.$$

The equilibrium payoffs are

$$\pi = \frac{m_h^2}{m_0} r.$$

Observe that, unlike in the case when $m_0 = 0$, the equilibrium supports have length r and not $2r$. Thus there is no aggressive strategy anymore; instead, the equilibrium is stabilized by competition over the m_0 customers. Customers who receive a rebate always buy at their home firm in equilibrium. This explains why a sufficiently large value of m_0 is needed to guarantee the existence of this equilibrium: if m_0 is too small, firms prefer to deviate to lower prices, attacking the opponent's home base.

This equilibrium with home-base customers always turning to their home firm brings to mind the equilibrium of Varian's (1980) model of sales where such a segmentation is exogenously assumed. In our model, however, this situation arises endogenously, and accordingly there are a number of notable differences. First, in Varian's model, firms would set infinite prices under an infinite willingness to pay. In contrast, in our model, the fact that the opponent may in principle attack stabilizes an equilibrium where firms mix over a bounded support. Moreover, in Varian's model, firms' equilibrium payoffs are independent of m_0 . Our model, however, has the surprising feature that the equilibrium payoffs decrease in m_0 . This is despite the fact that firms never earn negative payoffs from the m_0 customers. Intuitively, the reason is that a large value of m_0 leads to an alignment of the interests of the two firms and thus reduces their possibilities of segmentation.

In this light, another observation may be surprising: consider the above situation, but assume that firm 2 has an ex ante choice between setting the same rebate r as its opponent and setting a rebate of zero. The decision is observed before prices are chosen. Then it turns out that, for $m_0 > m_h/\beta$, where $\beta \approx 1.09$, firm 2 prefers to set a rebate of zero. The gains from facilitating price discrimination (by essentially merging m_0 and m_2) outweigh the loss from giving up a competitive advantage at the own home base.⁶

4. The generalized model

We now generalize the analysis by considerably weakening our assumptions on the demand function. A customer's demand depends on the lowest net price which he/she has to pay at either of the firms and is denoted by $X(\cdot)$. We impose the following assumptions on X : it is positive at least for small positive net prices and continuous and non-increasing in the net price. We also assume that the monopoly profits are bounded.⁷ We next distinguish two cases: in the first, all customers are homogeneous in the sense that all have the same rebate opportunities; in the second, the customers are heterogeneous, i.e., they have different rebate opportunities.

4.1. Homogeneous customers

Assume that the customers are homogeneous, i.e., $m_i > 0$ for exactly one $i \in \{0, 1, 2\}$. Then there is perfect competition in net prices, and hence the Bertrand Paradox arises: two firms are sufficient to yield the competitive outcome.

Proposition 5. *Suppose that the customers are homogeneous; then both firms earn zero profits.*

Next, we show that this is no longer true when the customers are heterogeneous.

⁶ The equilibrium payoffs for the case where one rebate equals zero can easily be calculated from Proposition 2.

⁷ This rules out equilibria à la Baye and Morgan (1999). They show (in a model without rebates) that, when the monopoly profits are unbounded, "any positive (but finite) payoff vector can be achieved in a symmetric mixed-strategy Nash equilibrium" (p. 59).

4.2. Heterogeneous customers

Assume that the customers are heterogeneous, i.e., $m_i = 0$ for at most one $i \in \{0, 1, 2\}$. This implies that the customers differ in the net prices they face. The next lemma states that in equilibrium no firm will charge a negative price. Loosely speaking, the reason is that a negative price leads to losses once something is sold. For a firm which offers a rebate, we get a stronger condition.

Lemma 1. *In any Nash equilibrium, no firm charges negative prices. A firm which offers a rebate charges prices well above zero.*

We next show that the Bertrand Paradox no longer arises.

Proposition 6. *In any Nash equilibrium, both firms earn positive expected profits.*

That is, when the customers are heterogeneous, competition is relaxed and firms earn positive expected profits. This also holds when only one firm offers a rebate. Generally, rebates make switching less attractive for customers. This segments the market and allows firms to earn profits. In contrast, without rebates or with rebates which can be used by all customers, the market does not get segmented, and firms earn zero profits; see Proposition 5.

When only one firm offers a rebate, its position in the price competition seems to be weak: when it attracts customers, it has to charge a sufficiently positive gross price to make no loss. In contrast, the competitor also makes no loss when it charges a price of zero. So why should a firm offer a rebate to some customers? The reason is that the competitor knows about the “weakness” of the rebate-offering firm and therefore sets a positive price in equilibrium. But, given this, the rebate-offering firm can target the potential rebate-receiving customers and obtain a positive expected profit.

So far, we have derived the characteristics of any Nash equilibrium. Yet we were silent in this section about equilibrium existence. Before we turn to this, we make an assumption which guarantees that playing very high prices is dominated.

Assumption 1. The demand function is elastic above a threshold price. Technically, $X(p)$ is such that there exists a \hat{p} so that $\varepsilon_{x,p} := -\frac{X'(p)}{X(p)/p} > 1 \forall p > \hat{p}$.

Sufficient conditions for Assumption 1 to hold are that for some price the demand function is elastic ($\varepsilon_{x,p} > 1$) and that the demand is log-concave (this implies, see Hermalin (2009), that $\varepsilon_{x,p}$ is increasing in p).

Lemma 2. *Under Assumption 1, playing prices above $\hat{p} + r_j$ is dominated for firm j .*

With the help of Lemma 2, we can establish the existence of a Nash equilibrium.

Proposition 7. *Under Assumption 1, for any tie-breaking rule, a Nash equilibrium exists.*

There is an alternative assumption to Assumption 1 which yields Lemma 2 and also Proposition 7. There is a choke price: $X(p) = 0 \forall p \geq \bar{p}$. Then prices above $\bar{p} + r_i$ are dominated for firm i .

Klemperer (1987a, Section 2) shows for an example that firms earn monopoly profits in their market segments. This result holds more generally.⁸

Proposition 8. *Suppose that $m_0 = 0, m_1, m_2 > 0$, and that there exists a monopoly price p^M . When the rebates r_1 and r_2 are sufficiently large, both firms earn monopoly profits in their market segment in equilibrium. An equilibrium in pure strategies supports this outcome. The same is true when there exists a choke price \bar{p} and $m_0, m_1, m_2 > 0$.*

Intuitively, when the rebates are high, no firm wants to attack the customers in the other firm’s home base. The reason is that such an attack would require setting a gross price which is low compared to the rebate the customers in the firm’s own home base get. Therefore, attacking would lead to a loss. This gives both firms the freedom to set gross prices such that customers pay net prices equal to the monopoly price. Thus the home base of firm i buys at firm i , and both firms earn monopoly profits in their market segment.

When there is a choke price which is low compared to the respective rebates, even the existence of customers who do not get rebates does not affect this result: firms still target only their home bases, because the high rebates make lower prices unattractive. Hence customers without rebate opportunities end up buying no product.

5. Endogenous rebates

Up to now we have concentrated on the price setting of the firms when the rebates are given. This approach may be a good description of the short-run behavior of firms where the rebate system is established and cannot be overturned. Additionally, in some industries such as aviation, several firms have a common rebate system. Then a firm can hardly change rebates when it decides about its prices.

Next, we offer some remarks on endogenous rebates. We keep the analysis brief and non-technical. Suppose that firms first set rebates simultaneously before they compete in prices. From Proposition 5, the following result is immediate.

Proposition 9. *That both firms set no rebate is not a subgame-perfect Nash equilibrium. It is also not subgame-perfect that both firms offer rebates to all customers.*

If neither firm sets a rebate, then both firms earn zero profit. This cannot be optimal because, by offering a rebate to some customers, a firm can earn a positive expected profit; see Proposition 6. The same arguments apply when firms offer rebates to all customers.

While our theoretical results allow for a rather explicit study of price setting under rebates, they are less well suited for studying how rebates come about. The reason is the following. For the demand functions for which explicit characterizations are possible, payoffs increase in rebates (or remain constant once both firms earn monopoly profits). We now use a simple numerical example of a slightly more general model to show that firms avoid setting “too high” rebates if there are customers who are able to receive both rebates or no rebate and if demand is decreasing continuously. This demonstrates that the mixed equilibria under moderate rebates studied above may persist in the case of endogenous rebates. For this, we consider the quadratically decreasing demand function

$$D(p) = \frac{10}{(1 + 3p)^2},$$

and assume that there is in total a mass 5 of customers – a mass 2 of customers who receive no rebates and masses 1 each of customers who receive both rebates or a rebate at either of the two firms. Firms first simultaneously set rebates, observe their rebates, and then set prices. To make the game tractable

⁸ The existence of a monopoly price is assumed in Proposition 8. One can easily show that Assumption 1 is sufficient for existence.

computationally, we discretize the choices of rebates and prices. We use a rather fine discretization for the price setting to ensure that the equilibrium payoffs from the price-setting stage are close to those from the continuous game. Concretely, firms can raise prices in 61 steps from 0 to 10.⁹ For rebate setting, we use a much rougher discretization, since our main aim here is to demonstrate that some rebates are too high to occur in equilibrium: the rebates are chosen from the set $\{0, 2, 4\}$.¹⁰ We calculate the equilibrium payoffs¹¹ of the price-setting game using the software package Gambit (McKelvey et al., 2010). This leads to the following rebate-setting game in the first stage.

	$r_2 = 0$	$r_2 = 2$	$r_2 = 4$
$r_1 = 0$	0, 0	2.5, 1.48	1.48, 0.05
$r_1 = 2$	1.48, 2.50	1.50, 1.50	1.48, 0.93
$r_1 = 4$	0.05, 1.48	0.93, 1.48	0.22, 0.22

This game has three Nash equilibria. They all have support only over the two lower rebates, 0 and 2. In the two pure equilibria, one firm sets a rebate of 2, earning a payoff of 1.48, while the other firm sets a rebate of 0, and earns a payoff of 2.50. There is also a symmetric mixed equilibrium where both firms mix over $\{0, 2\}$ with probabilities $(25/62, 37/62)$ and earn equilibrium payoffs of about 1.49.

Another reason that firms typically set only moderate rebates comes from the marketing literature. Brüggem et al. (2008) show that huge rebates are very harmful for a brand's image. More specifically, they find that “[e]very additional one percent of rebate is associated with a two point decline in the APEAL index [which is a measure of brand image]”. The recent change in pricing strategy by Europe's second largest car producer, namely PSA, is also motivated by the past experience that high rebates harm the brand image (cf. Financial Times Deutschland (2010)).

6. Concluding discussion

We have studied a price competition game in which customers are heterogeneous in the rebates they can get. We characterized the transition between competitive pricing (without rebates), mixed strategy equilibrium (for intermediate rebates), and monopoly pricing (for larger rebates). We observed that, in our mixed equilibrium, a firm's support consists of an aggressive part and a defensive part.

We observed that rebates lead to a segmentation of the market when the customers are heterogeneous. This segmentation has the effect that both firms earn positive expected profits. That is, by setting rebates, firms escape the Bertrand Paradox. Interestingly, for intermediate rebates, market segmentation may decrease in rebates. By means of a numerical study we showed that firms may optimally choose rebates of moderate size. We close with a discussion.

6.1. Welfare and customers' coordination problem

When there are no rebates, or when the customers are homogeneous, the net prices equal the marginal costs. Then the welfare optimum is obtained. With rebates and heterogeneous customers, at least some customers buy at positive net prices.

Hence, given a standard downward sloping demand function, the welfare optimum is no longer obtained.¹² Note that firms are in expectation better off (see Proposition 5 versus Proposition 6). Taken together, this implies that rebates deteriorate the customer welfare.

Customers face a coordination problem. They would collectively be better off when there are no rebates. This type of coordination is, however, not credible when there are many customers who cannot write contracts on whether or not they participate in rebate systems. First note that, when a customer has no mass, then he/she does not change the firms' pricing policies by participating or not participating in a rebate system. If he/she participates, he/she has the option to use the rebate and is therefore weakly better off than when he/she does not participate. There are cases where he/she is strictly better off. Therefore, each single customer is in expectation strictly better off by participating.

6.2. Miscellaneous

Heterogeneous demand. Note that the results obtained in Section 4 also hold when customer types have different demand functions: all proofs can be modified so that the demand function is type dependent as long as the demand functions fulfill the assumptions we made.

Discrimination. We assumed that firms cannot price discriminate. Technically, each firm has to offer a single gross price to all customers. Suppose now that firms can perfectly price discriminate. Then firms know what rebates a customer can get and are able to offer customer-specific gross prices. Hence, each customer can be thought of as an individual separate market. Because there is competition in prices, both firms will in equilibrium earn zero profits on each market. More specifically, in equilibrium, both firms offer each customer a gross price so that the net price equals the marginal production costs. Therefore, for the effectiveness of rebates, it is crucial that firms cannot discriminate.

More than two firms. Suppose that there are $N > 2$ firms. When the customers are homogeneous or at least two firms set no rebates, the Bertrand Paradox arises: it is an equilibrium that all firms set prices equal to their rebate and all firms obtain zero profits. Otherwise, the logic of Proposition 6 applies, and all firms earn positive expected profits.

Random distribution of coupons. Suppose that customers randomly receive rebate coupons: some might receive coupons from both firms, some from one firm, and others from no firm. This brings us to a situation like the one considered in the numerical simulation part of the paper. Generally, one can say that both firms must still earn positive expected profits in equilibrium. The line of argument is as before. First, both firms will only charge prices well above zero. Second, this gives both firms the opportunity to earn a positive profit by charging gross prices which are higher than their rebates.

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⁹ The upper bound 10 is sufficiently large so that no prices in its proximity are played in equilibrium for the choices of rebates we consider.

¹⁰ All these choices are made relatively arbitrary with the aim of obtaining a clear-cut example.

¹¹ The equilibria turn out to be unique except for the case $r_i = r_j = 0$, where the well-known (quantitatively minor) non-uniqueness of Bertrand competition with discrete prices arises.

¹² For the case in which there is constant demand, total welfare is constant for all prices at which customers buy. Nonetheless, rebates deteriorate the customer welfare.

Appendix. Proofs

Proof of Proposition 1. Special case of Proposition 8. □

Proof of Proposition 2. In order to identify an equilibrium candidate, we first assume that the equilibrium is indeed given by a function F_j that can be decomposed into an aggressive and a defensive strategy as sketched in the main text, i.e., $F_j = q_j A_j + (1 - q_j) D_j$. Furthermore, we postulate that the support of D_j corresponds to the support of A_i shifted upwards by r_j so that both distributions cover the same range of net prices for the customers in the home base of firm j . This is expressed by the system of equations

$$\bar{a}_i + r_j = \bar{d}_j \tag{9}$$

$$\bar{a}_j + r_i = \bar{d}_i \tag{10}$$

$$\underline{a}_i + r_j = \underline{d}_j \tag{11}$$

$$\underline{a}_j + r_i = \underline{d}_i. \tag{12}$$

Solving (2) and (3) for D_j and A_j , and shifting the argument, we get the following expressions for D_j and A_j :

$$D_j(p) = 1 - \frac{\pi_i - m_i(p - r_i - r_j)}{m_j(p - r_j)(1 - q_j)} \tag{13}$$

and

$$A_j(p) = \frac{1}{q_j} \left(1 - \frac{\pi_i}{m_i p} \right). \tag{14}$$

From these functions, it is easy to calculate $\underline{a}_j, \bar{a}_j, \underline{d}_j$, and \bar{d}_j as the prices where A_j and D_j take the values 0 and 1. This yields

$$\underline{a}_j = \frac{\pi_i}{m_i}, \quad \bar{a}_j = \frac{\pi_i}{m_i(1 - q_j)}$$

and

$$\underline{d}_j = \frac{\pi_i + m_i(r_i + r_j) + r_j m_j(1 - q_j)}{m_i + m_j(1 - q_j)}, \quad \bar{d}_j = \frac{\pi_i}{m_i} + r_i + r_j.$$

What remains to be done in order to pin down our equilibrium candidate is eliminating π_i, π_j, q_i , and q_j using the system of equations (9)–(12). Inserting the expressions for $\underline{a}_j, \bar{a}_j, \underline{d}_j$, and \bar{d}_j the system becomes

$$\frac{\pi_j}{m_j(1 - q_i)} = \frac{\pi_i}{m_i} + r_i \tag{15}$$

$$\frac{\pi_i}{m_i(1 - q_j)} = \frac{\pi_j}{m_j} + r_j \tag{16}$$

$$\frac{\pi_j}{m_j} + r_j = \frac{\pi_i + m_i(r_i + r_j) + r_j m_j(1 - q_j)}{m_i + m_j(1 - q_j)} \tag{17}$$

$$\frac{\pi_i}{m_i} + r_i = \frac{\pi_j + m_j(r_i + r_j) + r_i m_i(1 - q_i)}{m_j + m_i(1 - q_i)}. \tag{18}$$

Solving this system for π_i, π_j, q_i , and q_j yields the equilibrium candidate given in the statement of the proposition.¹³ We still need to check that this is well defined and that it is indeed an equilibrium. It is easy to see that A_i and D_i are indeed distribution functions, i.e., that they are monotonically increasing. (Then it follows by construction that they have the correct supports.) In order to check that q_i is indeed a probability, note that $q_i(m_i, m_j)$

only depends on the ratio m_i/m_j (and not on r_i and r_j). Thus it is sufficient to show that the univariate function $q_i(m_i, 1)$ only takes values in the interval $[0, 1]$. This is omitted here. To see that the supports of the defensive and aggressive strategies are adjacent, note that putting (16) and (17) together immediately yields

$$\bar{a}_j = \frac{\pi_i}{m_i(1 - q_j)} = \frac{\pi_i + m_i(r_i + r_j) + r_j m_j(1 - q_j)}{m_i + m_j(1 - q_j)} = \underline{d}_j.$$

By construction, firm j earns an expected payoff of π_j from playing a price in $[\underline{a}_j, \bar{d}_j]$. Thus, in order to show that we have indeed a Nash equilibrium, it remains to be shown that prices below \underline{a}_j or above \bar{d}_j are weakly dominated. Clearly, we can restrict attention to prices which are close enough to the supports of the equilibrium strategies to keep the two firms in competition: if firm i sets a price above $\bar{d}_j + r_i$ it obtains zero profits because all customers use firm j , and likewise there is a lower bound below which lowering the price even further will never lead to additional customers. Thus, consider firm i playing a price p (not too far) below \underline{a}_j while firm j plays its equilibrium strategy. It is easy to see that this leads to a payoff of

$$\pi_i(p) = m_i(p - r_i) + m_j p(1 - q_j A_j(p + r_j)) \tag{19}$$

for firm i . Now observe that, by multiplying (3) by m_j/m_i and shifting the argument p , we can conclude that, for some constant C_1 (which does not depend on p),

$$m_j(p + r_j)(1 - q_j A_j(p + r_j)) = C_1.$$

This allows us to rewrite (19) as

$$\pi_i(p) = C_1 + m_i(p - r_i) - m_j r_j(1 - q_j A_j(p + r_j)).$$

Thus $\pi_i(p)$ is an increasing function, which implies that playing \underline{a}_j dominates playing prices below it. We now turn to deviations to prices above \bar{d}_j . Playing such a price yields a payoff of

$$\pi_i(p) = m_i(p - r_i)(1 - q_j) D_j(p - r_i). \tag{20}$$

From (2), we can conclude that, for some constant C_2 ,

$$\frac{m_i^2}{m_j} p + m_i(p - r_i - r_j)(1 - q_j)(1 - D_j(p - r_i)) = C_2.$$

This yields

$$\pi_i(p) = C_2 - \frac{m_i^2}{m_j} p + m_i r_j(1 - q_j)(1 - D_j(p - r_i)).$$

Thus $\pi_i(p)$ is a decreasing function. This implies that playing \bar{d}_j dominates playing higher prices.

To conclude the proof of the proposition, we have to show that $A_i(p) = D_j(p + r_j)$. Note that by construction both $A_i(p)$ and $D_j(p + r_j)$ are probability distributions on $[\underline{a}_i, \bar{a}_i]$. Furthermore, by (13), $D_j(p + r_j)$ is given by

$$D_j(p + r_j) = 1 - \frac{\pi_i - m_i(p - r_i)}{m_j p(1 - q_j)} \quad \text{for } p \in [\underline{a}_i, \bar{a}_i].$$

Observe that both $D_j(p + r_j)$ and $A_i(p)$ are of the following form:

$$G(p) = \alpha - \frac{\beta}{p} \quad \text{for } p \in [\underline{a}_i, \bar{a}_i],$$

$G(\underline{a}_i) = 0$ and $G(\bar{a}_i) = 1$, where α and β are coefficients that do not depend on p . Then the boundary constraints $G(\underline{a}_i) = 0$ and $G(\bar{a}_i) = 1$ uniquely determine the values of the coefficients α and β . Thus $D_j(p + r_j)$ and $A_i(p)$ must be identical. □

¹³ There are three more solutions which do not correspond to equilibria. The reader who wants to verify that this is indeed a solution is strongly advised to utilize a computer algebra system such as Wolfram Mathematica.

Proof of Proposition 3. Case (i) is an immediate corollary of Proposition 2. The transition value r^* is calculated as the value of r for which $\bar{d} = 1 + r$. Likewise, it is easy to verify that the pure strategy equilibrium of Case (iii) is indeed an equilibrium. We can thus focus on Case (ii). An equilibrium candidate is constructed in a similar way as in the proof of Proposition 2: we still assume the existence of an aggressive and a defensive strategy whose respective supports differ by a shift by r . But in addition we make the restriction that $\bar{d} = 1 + r$ and allow for an atom of size $q^0 = 1 - q^A - q^D$ in $1 + r$. Here, q^A and q^D denote the probabilities of attacking and defending.¹⁴ Analogously to (2) and (3), we now get

$$\pi = 1 - q^A \tag{21}$$

$$\text{for } p = 1 + r,$$

$$\pi = (p - r)(1 - q^A A(p - r)) \tag{22}$$

$$\text{for } p \in [\underline{d}, \bar{d}], \text{ and}$$

$$\pi = p - r + p(1 - q^A - q^D D(p + r)) \tag{23}$$

for $p \in [\underline{a}, \bar{a}]$. Solving (22) and (23) for A and D , and using (21) to eliminate π , we get

$$D(p) = \frac{1}{q^D} \left(1 - q^A - \frac{1 - q^A - p + 2r}{p - r} \right)$$

and

$$A(p) = \frac{1}{q^A} \left(1 - \frac{1 - q^A}{p} \right).$$

Calculating the values where these functions become 0 or 1 yields the boundaries

$$\bar{a} = 1, \quad \underline{a} = 1 - q^A, \\ \bar{d} = \frac{r(1 - q^A - q^D) + 1 - q^A + 2r}{2 - q^A - q^D}, \quad \underline{d} = \frac{1 - q^A + 3r - rq^A}{2 - q^A}.$$

Solving the system of equations $\bar{a} + r = \bar{d}$ and $\underline{a} + r = \underline{d}$ yields the equilibrium values of q^A , q^D , and (through (21)) π . It is straightforward to verify that these strategies are well defined and that they interpolate between the strategies of Cases (i) and (iii). By construction, all prices in the support of the equilibrium strategy lead to the same payoff (given that the opponent plays his/her equilibrium strategy). Thus, to complete the proof it remains to be shown that prices outside the supports of A and D are dominated. Clearly, deviating to prices above \bar{d} leads to zero demand and is thus dominated. Playing prices between \bar{a} and \underline{d} attracts the same customers as playing a price of \underline{d} , and is thus dominated. Likewise, deviating to a price slightly (i.e., less than $\underline{d} - \bar{a}$) below \underline{a} is dominated, since it does not attract more customers than playing a price of \underline{a} . That deviating to even lower prices is dominated can be seen with an argument parallel to the one in the proof of Proposition 2. Likewise, the same argument as in the proof of Proposition 2 can be applied to show that $D(p + r) = A(p)$. \square

Proof of Proposition 4. Finding a symmetric equilibrium candidate is based on the conjecture that equilibrium supports have length r . This implies that customers who receive a rebate always buy at their home-base firm so that price competition is only over

¹⁴ For convenience, we drop the indices i and j throughout the proof. The analogous system of equations with possibly asymmetric payoffs and probabilities has the same symmetric equilibrium as its only solution which is a Nash equilibrium.

the m_0 customers. Denote thus by F an equilibrium price distribution function with support $[p, \underline{p} + r]$, and denote by $\pi(p)$ the payoff of a firm from playing price p given that the opponent mixes according to F . Clearly, we have

$$\pi(\underline{p}) = m_0 \underline{p} + m_h(\underline{p} - r) \quad \text{and} \quad \pi(\underline{p} + r) = m_h \underline{p}.$$

Setting $\pi(\underline{p}) = \pi(\underline{p} + r)$ immediately yields the desired values of \underline{p} and of the equilibrium payoffs π . The distribution function F can easily be calculated from

$$\pi = m_0 p(1 - F(p)) + m_h(p - r).$$

The proof that this is indeed an equilibrium under the given sufficient condition on m_0 is tedious but straightforward. It is thus omitted. The boundary case can be found as the case where firms are indifferent about marginally lowering their price at \underline{p} . \square

Proof of Lemma 1. First we prove that no firm charges negative prices in equilibrium.

Step (i). When only one firm charges possibly negative prices, this firm obtains a loss when it plays such a negative price since at least the customer which get no rebate from the other firm buy from this firm. This cannot be optimal since zero profits can always be guaranteed.

Step (ii). When two firms possibly charge negative prices, customers will buy for sure when at least one firm indeed charges a negative price. Hence, at least one firm will sell a positive amount with positive probability when it charges a negative price. This firm's expected profit from charging this price is therefore negative. This cannot be optimal since zero profits can always be guaranteed.

Next, we prove that a firm which offers a rebate charges prices well above zero. Suppose that firm 1 offers a rebate $r_1 > 0$. From before, we know that firm 2 will not charge negative prices. Hence, when firm 1 charges prices $\in [0, r_1]$, at least the customers in its home base buy from it. Firm 1's expected profit increases when it gets more likely that also other customers buy from it. Suppose that also the other customers buy with probability 1. Then

$$\pi_1(p_1) = (p_1 - r_1)m_1 X(p_1 - r_1) + p_1 m_2 X(p_1).$$

We denote the total mass of customers by

$$m := m_0 + m_1 + m_2.$$

As X is non-increasing and $m_{-1} = m - m_1$, we have

$$\begin{aligned} \pi_1(p_1) &\leq (p_1 - r_1)m_1 X(p_1 - r_1) + p_1(m - m_1)X(p_1 - r_1) \\ &= ((p_1 - r_1)m_1 + p_1(m - m_1))X(p_1 - r_1) \end{aligned}$$

for all $p_1 \geq 0$. Hence, for all $p_1 \in (-\infty, r_1 m_1/m)$, we must have $\pi_1(p_1) < 0$. These prices are clearly dominated. \square

Proof of Proposition 6. Suppose that firm 1 offers a rebate. From Lemma 1, we know that then $p_1 \geq r_1 m_1/m$, where $m := m_0 + m_1 + m_2$.

Case 1: firm 2 offers no rebates. Firm 2 can set $p_2 \nearrow r_1 m_1/m$. Then all customers who do not get a rebate from firm 1 will buy from firm 2, when they buy. When they buy, firm 2 earns a nontrivial profit. When they do not buy for this price, firm 2 can lower the price so that it sells a positive amount and earns a nontrivial positive profit.

Case 2: firm 2 offers a rebate. When firm 2 sets the price $p_2 \nearrow r_1 m_1/m + r_2$, it gets all customers in its home base, when they buy at all.

For both cases we have shown that there exists a lower bound on firm 2's expected profit which is well above zero. Call this lower bound π_2 . Next, we have to prove that also firm 1 earns an expected profit well above zero. Since for p_2 near zero firm 2's expected profit is below π_2 , firm 2 must charge prices well above zero in equilibrium. This enables firm 1 to earn a nontrivial positive profit. The arguments correspond to the ones of Case 2 above. \square

Proof of Lemma 2. First, note that, when $\varepsilon_{x,p} > 1$, the revenue $R(p) = pX(p)$ is decreasing in p . Hence, conditional on the customers from firm i 's home base buying from firm j , firm j 's profit from this customer segment is decreasing in the net price when the net price exceeds \hat{p} . Moreover, the probability that customers from firm i 's home base buy from firm j is weakly decreasing in p_j for every price-setting strategy of firm i . We next have to distinguish two cases. Suppose that firm j sets a price $\check{p} > \hat{p} + r_j$.

Case 1: the expected profit of firm j is positive for \check{p} . The price \check{p} is dominated by the price $\hat{p} + r_j$ because then (i) the profit from selling to each customer segment is positive for \check{p} and for $\hat{p} + r_j$, and (ii) from the arguments before we know that setting $\hat{p} + r_j$ instead of \check{p} leads to a weakly higher probability that customers buy and to higher revenues and profits, conditional that customers buy from firm j .

Case 2: the expected profit of firm j is non-positive for \check{p} . From Proposition 6, we know that there are prices so that the firm earns a positive expected profit. Hence, playing gross prices exceeding $\hat{p} + r_j$ is dominated. \square

Proof of Proposition 7. Denote our game by G . Recall that we assumed monopoly payoffs and thus monopoly prices to be bounded. Denote by G' the modified game in which firms' pricing strategies are restricted to lie in $[0, u_j]$, where $u_j = \hat{p} + \max\{r_i, r_j\}$. From Lemmas 1 and 2, we know that playing prices outside $[0, u_j]$ is strictly dominated in G . Thus any Nash equilibrium of G' is also a Nash equilibrium of G . Define the set S^* by

$$S^* = [0, u_1] \times [0, u_2] \setminus \{(s_1, s_2) | s_1 + r_1 = s_2 \text{ or } s_2 + r_2 = s_1\}.$$

S^* lies dense in the set of actions $[0, u_1] \times [0, u_2]$. Furthermore, the payoffs are bounded and continuous in S^* . Thus by Simon and Zame (1990, p. 864), there exists a tie-breaking rule in G' for which a Nash equilibrium exists. Now observe that tie breaking occurs in any equilibrium with probability 0: suppose that tie breaking occurs with positive probability. This can only be due to both firms setting atoms in a way that a tie occurs (i.e., at distance r_1 or r_2). By Proposition 6, the supports of both players' equilibrium strategies must be bounded away from 0. Hence at least one firm has an incentive to slightly shift its atom downwards. Thus we can conclude that G' has a Nash equilibrium for any tie-breaking rule. This Nash equilibrium is also a Nash equilibrium of G . \square

Proof of Proposition 8. We first show that the prices $p_i = p^M + r_i$ and $p_j = p^M + r_j$ form a Nash equilibrium for sufficiently large r_i and r_j . Since these strategies imply that each firm earns monopoly profits from its market segment, a deviation can only be profitable if it attracts additional customers from the other firm's segment. Thus it is sufficient to consider deviations to prices $p \in [0, p^M]$. Suppose that r_i is sufficiently large so that $p^M - r_i < 0$. Then firm i 's profit from deviating to a price $p \in [0, p^M]$ can be bounded from above as follows:

$$m_i(p - r_i)X(p - r_i) + m_j pX(p) < m_i(p^M - r_i)X(p^M) + m_j p^M X(p^M),$$

since $X(p^M) \leq X(p - r_i)$ and since $pX(p) \leq p^M X(p^M)$. If r_i is sufficiently large, the upper bound becomes negative, so deviations cannot be profitable.

So far we have shown that for sufficiently high rebates there exists an equilibrium where both firms earn monopoly profits in their market segment. Now we show that this has to be true in any equilibrium. From Lemma 1, we know that no firm will charge negative prices. From before, we know that for sufficiently high rebates a firm obtains a loss if it charges a price $p \in [0, p^M]$. Therefore, $p_1, p_2 > p^M$ in any equilibrium. Hence, by charging a

price of $p^M + r_i$, firm i can guarantee a profit of at least $m_i p^M X(p^M)$. Therefore, in equilibrium, the expected profit of firm i must be at least $m_i p^M X(p^M)$. This holds for both firms. Therefore, in an equilibrium, the sum of both firms' expected profits is at least $(m_1 + m_2)p^M X(p^M)$. By the definition of the monopoly profit, the maximum sum of profits is $(m_1 + m_2)p^M X(p^M)$. All this is compatible only if firm 1 earns an expected profit of $m_1 p^M X(p^M)$ and firm 2 earns an expected profit of $m_2 p^M X(p^M)$. That is, there can only be equilibria in which firms earn expected profits equal to the monopoly profits in their market segment.

Next, we prove the final part of the proposition which considers the case where $m_0, m_1, m_2 > 0$, and where a choke price exists. The proof is similar to that before, and is therefore only sketched. First, when the rebates are sufficiently high a firm obtains a loss when it charges a price below the choke price. Second, therefore in equilibrium the prices are above the choke price. Third, this implies that, in equilibrium, customers without rebate opportunities do not buy. Fourth, therefore customers without rebate opportunities can be ignored, and the proof for the case $m_0 = 0$ applies. \square

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